Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null-controllability in cylindrical domains

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SHARP ESTIMATES OF THE ONE-DIMENSIONAL BOUNDARY CONTROL COST FOR PARABOLIC SYSTEMS AND APPLICATION TO THE N-DIMENSIONAL BOUNDARY NULL-CONTROLLABILITY IN CYLINDRICAL DOMAINS

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Abstract. In this paper we consider the boundary null-controllability of a system of n parabolic equations on domains of the form $\Omega = (0,\pi) \times \Omega_2$ with Ω_2 a smooth domain of $\mathbb{R}^{N-1}$, $N > 1$. When the control is exerted on $\{0\} \times \omega_2$ with $\omega_2 \subset \Omega_2$, we obtain a necessary and sufficient condition that completely characterizes the null-controllability. This result is obtained through the Lebeau-Robbiano strategy and require an upper bound of the cost of the one-dimensional boundary null-control on $(0,\pi)$. This latter is obtained using the moment method and it is shown to be bounded by $Ce^{C/T}$ when T goes to 0^+.

Key words. Parabolic systems, Boundary Controllability, Biorthogonal families, Kalman Rank condition.

AMS subject classifications. 93B05, 93C05, 35K05.

1. Introduction. The controllability of systems of n partial differential equations by $m < n$ controls is a relatively recent subject. We can quote [LZ98], [dT00], [BN02] among the first works. More recently in [AKBDGB09b], with fine tools of partial differential equations, the so-called Kalman rank condition, which characterizes the controllability of linear systems in finite dimension, has been generalized in view of the distributed null-controllability of some classes of linear parabolic systems. On the other hand, while for scalar problems the boundary controllability is known to be equivalent to the distributed controllability, it has been proved in [FCGBdT10] that this is no more the case for systems. This reveals that the controllability of systems is much more subtle. In [AKBGBdT12], it is even showed that a minimal time of control can appear if the diffusion is different on each equation, which is quite surprising for a system possessing an infinite speed of propagation. It is important to emphasize that the previous quoted results concerning the boundary controllability were established in space dimension one. They used the moment method, generalizing the works of [FR71, FR75] concerning the boundary controllability of the one-dimensional scalar heat equation. We refer to [AKBGBdT11b] for more details and a survey on the controllability of parabolic systems.

In higher space dimension the boundary controllability of parabolic systems remains widely open and it is the main purpose of this article to give some partial answers. To our knowledge, the only results on this issue are the one of [ABL12] and [AB12]. Let us also mention [Oli13] for related questions for the approximate controllability problem. In [ABL12, AB12] the results for parabolic systems are deduced from the study of the boundary control problem of two coupled wave equations using transmutation techniques. As a result there are some geometric constraints on the control domain. We will see that this restriction is not necessary.

In the present work, we focus on the boundary null-controllability of the following n coupled parabolic equations by m controls in dimension $N > 1$

$$\begin{cases} \partial_t y = \Delta y + Ay & \text{in } (0,T) \times \Omega, \\ y = 1_\gamma B v & \text{on } (0,T) \times \partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

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in the case where the domain Ω has a Cartesian product structure

$$\Omega = \Omega_1 \times \Omega_2,$$

where $\Omega_i \subset \mathbb{R}^{N_i}$, $i = 1, 2$ are bounded open regular domains. In (1.1), $T > 0$ is the control time, the non-empty relative subset $\gamma \subset \partial\Omega$ is the control domain, y is the state, y_0 is the initial data, $A \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{M}_{n\times m}(\mathbb{C})$ are constant matrices and v is the boundary control.

Under appropriate assumptions we show that the controllability of System (1.1) is reduced to the controllability of the same system posed on Ω_1 (see Theorem 1.3 below). The proof is based on the method of Lebeau-Robiano [LR95]. This strategy (already used in a different framework in [BDR07]) requires an estimate of the cost of the N_1-dimensional control with respect to the control time when $T \to 0^+$.

In a second part, we establish that the cost of the one-dimensional null-control on $(0, T)$ is bounded by $Ce^{C/T}$, for some $C > 0$, as $T \to 0^+$ (see Theorem 1.4 below). This is the second main result of this paper and this also shows that our first result above can be applied at least in the case $N_1 = 1$. The demonstration of this result follows the approach of [FR71] and [Mil04] (for the scalar case). It requires to take back the proofs contained in [AKBGBdT11a]. In the scalar case, [Sei84] (see also [FCZ00]) gave a similar estimate of the cost of the boundary control of the heat equation, which is known to be optimal thanks to the work [Güi85].

Note finally, that the extension of the present results to more general domains Ω in $\mathbb{R}^N$ as well as the study of the case with different diffusion coefficient on each equation remain open problems.

1.1. Reminders and notations. Let us first recall that System (1.1) is well-posed in the sense that, for every $y_0 \in H^{-1}(\Omega)^n$ and $v \in L^2(0, T; L^2(\partial\Omega)^m)$, there exists a unique solution $y \in C^0([0, T]; H^{-1}(\Omega)^n) \cap L^2(0, T; L^2(\Omega)^n)$, defined by transposition. Moreover, this solution depends continuously on the initial data y_0 and the control v. More precisely,

$$\|y\|_{C^0([0,T];H^{-1}(\Omega)^n)} \leq Ce^{CT}\left(\|y_0\|_{H^{-1}(\Omega)^n} + \|v\|_{L^2(0,T;L^2(\partial\Omega)^m)}\right), \tag{1.2}$$

where here and all along this work $C > 0$ denotes a generic positive constant that may change line to line but which does not depend on T nor y_0. We shall also use sometimes the notations C', C'', and so on.

Let us now precise the concept of controllability we will deal with in this paper. We say that System (1.1) is null-controllable at time T if for every $y_0 \in H^{-1}(\Omega)^n$, there exists a control $v \in L^2(0, T; L^2(\partial\Omega)^m)$ such that the corresponding solution y satisfies

$$y(T) = 0.$$

In such a case, it is well-known that there exists $C_T > 0$ such that

$$\|v\|_{L^2(0,T;L^2(\partial\Omega)^m)} \leq C_T\|y_0\|_{H^{-1}(\Omega)^n}, \quad \forall y_0 \in H^{-1}(\Omega)^n. \tag{1.3}$$

The infimum of the constants C_T satisfying (1.3) is called the cost of the null-control at time T.

Remark 1. *Even if it means replacing $y(t)$ by $e^{-\mu t}y(t)$ and A by $A - \mu$, with $\mu > 0$, we can assume without loss of generality that the matrix A is stable: all its eigenvalues have a negative real part.*

Finally, let us recall the well-known duality between controllability and observability.

Theorem 1.1. *Let E be a closed subspace of $H^1_0(\Omega)^n$ and set $E^{-1} = -\Delta E \subset H^{-1}(\Omega)^n$. Let us denote Π_E (resp. $\Pi_{E^{-1}}$) the orthogonal projection on E (resp. E^{-1}). Let $C_T > 0$ be fixed. For every $y_0 \in E^{-1}$ there exists a control $v \in L^2(0, T; L^2(\partial\Omega)^m)$ such that*

$$\begin{cases} \Pi_{E^{-1}}y(T) = 0, \\ \|v\|_{L^2(0,T;L^2(\partial\Omega)^m)} \leq C_T\|y_0\|_{H^{-1}(\Omega)^n}, \end{cases}$$

where y is the corresponding solution to (1.1), if and only if

$$\|\Pi_E z(0)\|^2_{H^1_0(\Omega)^n} \leq C_T^2 \int_0^T \|1_\gamma B^*\partial_n z(t)\|^2_{L^2(\partial\Omega)^m}dt, \quad \forall z_T \in E,$$

where z is the solution to the adjoint system

$$\left\{\begin{array}{rcll} -\partial_t z &=& \Delta z + A^* z & \text{in } (0,T)\times\Omega, \\ z &=& 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0) &=& z_T & \text{in } \Omega. \end{array}\right. \tag{1.4}$$

Notations. We gather here some standard notations that we shall use all along this paper. For any real numbers $a<b$ we denote $[\![a,b]\!] = [a,b]\cap\mathbb{Z}$. For $z\in\mathbb{C}$, $\Re(z)$ and $\Im(z)$ denote the real and imaginary part of z. Finally, $x\in\mathbb{R}\mapsto\lfloor x\rfloor\in\mathbb{Z}$ denotes the floor function.

1.2. Main results.

1.2.1. Boundary controllability for a multidimensional parabolic system. The first main achievement of this work is the following.

THEOREM 1.2. *Let $\omega_2\subset\Omega_2$ be a non-empty open subset and take $\Omega_1=(0,\pi)$. Then, System (1.1) is null-controllable at time T on $\gamma=\{0\}\times\omega_2$ if and only if*

$$\operatorname{rank}(B_k|A_kB_k|A_k^2B_k|\cdots|A_k^{nk-1}B_k) = nk, \quad \forall k\geq 1, \tag{1.5}$$

where we have introduced the notations

$$A_k = \begin{pmatrix} -\lambda_1+A & 0 & \cdots & \cdots & 0 \\ 0 & -\lambda_2+A & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -\lambda_k+A \end{pmatrix} \in\mathcal{M}_{nk}(\mathbb{C}), \quad B_k = \begin{pmatrix} B \\ B \\ \vdots \\ \vdots \\ B \end{pmatrix} \in\mathcal{M}_{nk\times m}(\mathbb{C}). \tag{1.6}$$

One may think to a cylindrical domain where the control domain is a subset of the top or bottom face (see Figure 1.1).

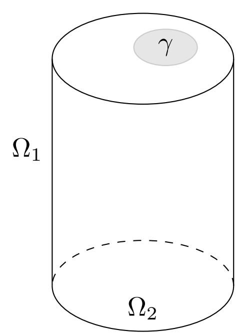


FIG. 1.1. *Typical geometric situation*

This result will be otained as a corollary of some other theorems that are important results too. The first one is the following and it should be connected with [Fat75] and [Mil05].

THEOREM 1.3. *Let $\gamma_1\subset\partial\Omega_1$ be a non-empty relative subset. Assume that the following N_1-dimensional system*

$$\left\{\begin{array}{rcll} \partial_t y^1 &=& \Delta_{x_1} y^1 + Ay^1 & \text{in } (0,T)\times\Omega_1, \\ y^1 &=& 1_{\gamma_1} Bv^1 & \text{on } (0,T)\times\partial\Omega_1, \\ y^1(0) &=& y^1_0 & \text{in } \Omega_1, \end{array}\right. \tag{1.7}$$

is null-controllable for any time $T > 0$, with in addition the following bound for the control cost $C_T^{\Omega_1}$

$$C_T^{\Omega_1} \leq Ce^{C/T}, \quad \forall T > 0. \tag{1.8}$$

Then, for any non-empty open set $\omega_2 \subset \Omega_2$, the N-dimensional System (1.1) is null-controllable at any time $T > 0$ on the control domain $\gamma = \gamma_1 \times \omega_2$.

REMARK 2. *The converse of Theorem 1.3 also holds. More precisely, if the N-dimensional System (1.1) is null-controllable at time T, then the N_1-dimensional System (1.7) is also null-controllable at time T. This can be proved using a Fourier decomposition in the direction of Ω_2.*

It is worth mentioning that, such a decomposition also shows that, when $\omega_2 = \Omega_2$, the proof of Theorem 1.3 is much simpler and it does not need the control cost estimate (1.8). Moreover, the domain Ω_2 can even be unbounded in this case.

1.2.2. Estimate of the control cost for a 1D boundary controllability problem. The second result of this paper provides an important example where Theorem 1.3 can be successfully applied.

More precisely, we show that the assumption (1.8) on the short time behavior of the control cost actually holds in the 1D case for the following system if we assume the rank condition (1.5)

$$\begin{cases} \partial_t y = \partial_{xx}^2 y + Ay & \text{in } (0,T) \times (0,\pi), \\ y(t,0) = Bv(t), \ y(t,\pi) = 0 & \text{in } (0,T), \\ y(0) = y_0 & \text{in } (0,\pi). \end{cases} \tag{1.9}$$

We recall that it has been established in [AKBGBdT11a] that System (1.9) is null-controllable at time $T > 0$ if and only if the rank condition (1.5) holds.

However, in the above-mentioned reference, no estimate on the control cost is provided. This is the next goal of the present paper, to give a more precise insight into the proof of the controllability result for System (1.9) that allows a precise estimate of the control cost as a function of T.

THEOREM 1.4. *Assume that the rank condition (1.5) holds. Then, for every $T > 0$ and $y_0 \in H^{-1}(0,\pi)^n$ there exists a null-control $v \in L^2(0,T)^m$ for System (1.9) which, in addition, satisfies*

$$\|v\|_{L^2(0,T)^m} \leq Ce^{C/T} \|y_0\|_{H^{-1}(0,\pi)^n}.$$

This theorem, combined with Theorem 1.3 and Remark 2 give a proof of Theorem 1.2.

1.2.3. Bounds on biorthogonal families of exponentials. The proof of Theorem 1.4 is mainly based on the existence of a suitable biorthogonal family of time-dependent exponential functions. The construction provided in [AKBGBdT11a] does not allow to estimate the control cost. That is the reason why we propose here a slightly different approach which is the key to obtain the factor $e^{C/T}$. This abstract result, which is interesting in itself and potentially useful in other situations, can be formulated as follows.

THEOREM 1.5. *Let $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ be a sequence of complex numbers with the following properties*

($\mathcal{H}_1$) $\Lambda_k \neq \Lambda_n$ *for all $k, n \in \mathbb{N}$ with $k \neq n$.*

($\mathcal{H}_2$) $\Re(\Lambda_k) > 0$ *for every $k \geq 1$.*

($\mathcal{H}_3$) *For some $\beta > 0$,*

$$|\Im(\Lambda_k)| \leq \beta \sqrt{\Re(\Lambda_k)}, \quad \forall k \geq 1.$$

($\mathcal{H}_4$) $\{\Lambda_k\}_{k\geq 1}$ *is non-decreasing in modulus*

$$|\Lambda_k| \leq |\Lambda_{k+1}|, \quad \forall k \geq 1.$$

($\mathcal{H}_5$) $\{\Lambda_k\}_{k\geq 1}$ *satisfies the following gap condition: for some $\rho, q > 0$,*

$$\begin{cases} |\Lambda_k - \Lambda_n| \geq \rho \left|k^2 - n^2\right|, & \forall k, n : |k-n| \geq q. \\ \inf_{k\neq n:|k-n|<q} |\Lambda_k - \Lambda_n| > 0. \end{cases}$$

($\mathcal{H}_6$) *For some $p, \alpha > 0$,*

$$\left|p\sqrt{r} - \mathcal{N}(r)\right| \leq \alpha, \quad \forall r > 0, \tag{1.10}$$

where $\mathcal{N}$ is the counting function associated with the sequence $\{\Lambda_k\}_{k\geq 1}$, that is the function defined by

$$\mathcal{N}(r) = \#\left\{k : |\Lambda_k| \leq r\right\}, \quad \forall r > 0. \tag{1.11}$$

Then, there exists $T_0 > 0$ such that, for every $\eta \geq 1$ and $0 < T < T_0$, we can find a family of $\mathbb{C}$-valued functions

$$\{\varphi_{k,j}\}_{k\geq 1, j\in[\![0,\eta-1]\!]} \subset L^2(-T/2, T/2)$$

biorthogonal [1] *to $\{e_{k,j}\}_{k\geq 1, j\in[\![0,\eta-1]\!]}$, where for every $t \in (-T/2, T/2)$,*

$$e_{k,j}(t) = t^j e^{-\Lambda_k t},$$

with in addition

$$\|\varphi_{k,j}\|_{L^2(-T/2,T/2)} \leq Ce^{C\sqrt{\Re(\Lambda_k)} + \frac{C}{T}}, \tag{1.12}$$

for any $k \geq 1$, $j \in [\![0, \eta-1]\!]$.

2. Boundary null-controllability on product domains.

2.1. Settings and preliminary remarks. Let $\lambda_j^{\Omega_1}$ (resp. $\lambda_j^{\Omega_2}$), $j \geq 1$, be the Dirichlet eigenvalues of the Laplacian on Ω_1 (resp. Ω_2), and let $\phi_j^{\Omega_1}$ (resp. $\phi_j^{\Omega_1}$) be the corresponding normalized eigenfunction.

Let us introduce the (closed) subspaces of $H_0^1(\Omega)^n$ on which we will establish the partial observability later on (section 2.2)

$$E_J = \left\{ \sum_{j=1}^{J} \left\langle u, \phi_j^{\Omega_2} \right\rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2} \,\middle|\, u \in H_0^1(\Omega)^n \right\} \subset H_0^1(\Omega)^n, \quad J \geq 1,$$

where the notation $\sum_{j=1}^{J} \langle u, \phi_j^{\Omega_2}\rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2}$ is used to mean the function

$$(x_1, x_2) \in \Omega \longmapsto \sum_{j=1}^{J} \langle u(x_1, \cdot), \phi_j^{\Omega_2}\rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2}(x_2).$$

We then define the "dual" spaces of E_J

$$E_J^{-1} = -\Delta E_J \subset H^{-1}(\Omega)^n, \quad J \geq 1.$$

Let us recall that we denote by Π_{E_J} (resp. $\Pi_{E_J^{-1}}$) the orthogonal projection in $H_0^1(\Omega)^n$ (resp. $H^{-1}(\Omega)^n$) onto E_J (resp. E_J^{-1}). It is not difficult to see that we have the relation $\Pi_{E_J^{-1}}(-\Delta u) = -\Delta \Pi_{E_J} u$ for any $u \in H_0^1(\Omega)^n$.

LEMMA 2.1. *For any $u \in H_0^1(\Omega)^n$, we have*

$$u = \sum_{j=1}^{+\infty} \langle u, \phi_j^{\Omega_2}\rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2}.$$

It follows from this lemma that $\Pi_{E_J} u = \sum_{j=1}^{J} \langle u, \phi_j^{\Omega_2}\rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2}$ for any $u \in H_0^1(\Omega)^n$.

[1] that is $\langle \varphi_{k,j}, e_{l,\nu}\rangle_{L^2(-T/2,T/2)} = \int_{-T/2}^{T/2} \varphi_{k,j}(t)\overline{e_{l,\nu}}(t)\, dt = \delta_{kl}\delta_{j\nu}$.

Proof of Lemma 2.1. Let us show that the sequence $\{S_J u\}_{J\geq 1}$ defined by

$$S_J u = \sum_{j=1}^{J} \langle u, \phi_j^{\Omega_2} \rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2},$$

is a Cauchy sequence of $H_0^1(\Omega)^n$. For any $J > K \geq 1$ we have

$$\begin{aligned} \|S_J u - S_K u\|^2_{H_0^1(\Omega)^n} &= \left\| \sum_{j=K+1}^{J} \left\langle u, \phi_j^{\Omega_2} \right\rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2} \right\|^2_{H_0^1(\Omega)^n} \\ &= \sum_{j=K+1}^{J} \left\| \left\langle u, \phi_j^{\Omega_2} \right\rangle_{L^2(\Omega_2)} \right\|^2_{H_0^1(\Omega_1)^n} + \sum_{j=K+1}^{J} \lambda_j^{\Omega_2} \left\| \left\langle u, \phi_j^{\Omega_2} \right\rangle_{L^2(\Omega_2)} \right\|^2_{L^2(\Omega_1)^n} \end{aligned}$$

Using Lebesgue's dominated convergence theorem it is not difficult to see that these terms go to zero as $J, K \to +\infty$. As a result $S_J u \xrightarrow[J\to+\infty]{H_0^1} v$ for some $v \in H_0^1(\Omega)^n$. In particular, $\langle v, \phi_k^{\Omega_1} \phi_j^{\Omega_2} \rangle_{L^2(\Omega)} = \langle u, \phi_k^{\Omega_1} \phi_j^{\Omega_2} \rangle_{L^2(\Omega)}$ for every $j, k \geq 1$, and it follows that $v = u$.

2.2. Partial observability. One of the key points to make use of the Lebeau-Robbiano strategy is the estimate of the cost of the partial observabilities on the approximation subspaces. This will be used for the active control phase.

Proposition 2.2. *Let Ω_2 be of class C^2. Assume that System (1.7) is controllable at time T with cost $C_T^{\Omega_1}$. Then,*

$$\|\Pi_{E_J} z(0)\|^2_{H_0^1(\Omega)^n} \leq C (C_T^{\Omega_1})^2 e^{C\sqrt{\lambda_J^{\Omega_2}}} \int_0^T \|1_{\gamma_1 \times \omega_2} B^* \partial_n z(t)\|^2_{L^2(\partial\Omega)^m} dt, \quad \forall z_T \in E_J, \tag{2.1}$$

where z is the solution to the adjoint system (1.4).

By Theorem 1.1 we deduce that

Corollary 2.3. *For every $J \geq 1$ and $y_0 \in E_J^{-1}$, there exists a control $v(y_0) \in L^2(0,T; L^2(\partial\Omega)^m)$ with*

$$\|v(y_0)\|_{L^2(0,T;L^2(\partial\Omega)^m)} \leq C\, (C_T^{\Omega_1}) e^{C\sqrt{\lambda_J^{\Omega_2}}} \|y_0\|_{H^{-1}(\Omega)^n}, \tag{2.2}$$

such that the solution y to system (1.1) *satisfies*

$$\Pi_{E_J^{-1}} y(T) = 0.$$

Proof of Proposition 2.2. Let $z_T \in E_J$ so that

$$z_T(x_1, x_2) = \sum_{j=1}^{J} z_T^j(x_1) \phi_j^{\Omega_2}(x_2),$$

for some $z_T^j \in H_0^1(\Omega_1)^n$. Let z be the solution of (1.4), the adjoint system of (1.1), associated with z_T. Thus,

$$z(t, x_1, x_2) = \sum_{j=1}^{J} z^j(t, x_1) \phi_j^{\Omega_2}(x_2),$$

where z^j is the solution to

$$\begin{cases} -\partial_t z^j &= \left(\Delta_{x_1} - \lambda_j^{\Omega_2}\right) z^j + A^* z^j & \text{in } (0,T) \times \Omega_1, \\ z^j &= 0 & \text{on } (0,T) \times \partial\Omega_1, \\ z^j(T) &= z_T^j & \text{in } \Omega_1. \end{cases}$$

Note that $\Pi_{E_J} z(0) = z(0)$. A computation of $\|z(0)\|^2_{H^1_0(\Omega)^n}$ gives

$$\|z(0)\|^2_{H^1_0(\Omega)^n} = \sum_{j=1}^{J} \left\|z^j(0)\right\|^2_{H^1_0(\Omega_1)^n} + \sum_{j=1}^{J} \lambda_j^{\Omega_2} \left\|z^j(0)\right\|^2_{L^2(\Omega_1)^n}.$$

Using the Poincaré inequality we obtain,

$$\|z(0)\|^2_{H^1_0(\Omega)^n} \leq C\lambda_J^{\Omega_2} \sum_{j=1}^{J} \left\|z^j(0)\right\|^2_{H^1_0(\Omega_1)^n}. \tag{2.3}$$

Observe now that $z^j(t) = e^{-(T-t)\lambda_j^{\Omega_2}} \psi(t)$, where ψ is the solution to the adjoint system of (1.7) associated with z^j_T. Thus, using the assumption that (1.7) is controllable with cost $C_T^{\Omega_1}$, we obtain by Theorem 1.1 that

$$\left\|z^j(0)\right\|^2_{H^1_0(\Omega_1)^n} \leq \left(C_T^{\Omega_1}\right)^2 \int_0^T \left\|1_{\gamma_1} B^* \partial_{n_1} z^j(t)\right\|^2_{L^2(\partial\Omega_1)^m} dt,$$

where n_1 denotes the unit outward normal vector of Ω_1. Combined to (2.3), this gives

$$\|z(0)\|^2_{H^1_0(\Omega)^n} \leq C\left(C_T^{\Omega_1}\right)^2 \lambda_J^{\Omega_2} \int_0^T \sum_{j=1}^{J} \left\|1_{\gamma_1} B^* \partial_{n_1} z^j(t)\right\|^2_{L^2(\partial\Omega_1)^m} dt.$$

Let us denote by B_k the kth column of B. Applying the Lebeau-Robbiano's spectral inequality [LR95] (see also [LR07, Section 3.A] [2])

$$\sum_{j=1}^{J} |a_j|^2 \leq Ce^{C\sqrt{\lambda_J^{\Omega_2}}} \int_{\omega_2} \left| \sum_{j=1}^{J} a_j \phi_j^{\Omega_2}(x_2) \right|^2 dx_2$$

to the sequence of scalars $a_j = B_k^* \partial_{n_1} z^j(t, \sigma_1)$, $\sigma_1 \in \partial\Omega_1$ being fixed, and summing over $1 \leq k \leq m$, this gives

$$\sum_{j=1}^{J} \left|B^* \partial_{n_1} z^j(t, \sigma_1)\right|^2_{\mathbb{C}^n} \leq Ce^{C\sqrt{\lambda_J^{\Omega_2}}} \int_{\omega_2} \left| \sum_{j=1}^{J} B^* \partial_{n_1} z^j(t, \sigma_1) \phi_j^{\Omega_2}(x_2) \right|^2_{\mathbb{C}^n} dx_2.$$

To conclude it only remains to integrate over γ_1 and observe that

$$n(\sigma) = \begin{pmatrix} n_1(\sigma_1) \\ 0 \end{pmatrix} \text{ for } \sigma = (\sigma_1, x_2) \in \partial\Omega_1 \times \Omega_2.$$

2.3. Dissipation along the direction Ω_2. The other point of the Lebeau-Robbiano strategy relies on the natural dissipation of the system when no control is exerted (the passive phase). For our purpose, we need an exponential dissipation in the direction Ω_2.

Proposition 2.4. *If there is no control on (t_0, t_1) (i.e. $v = 0$ on (t_0, t_1)) and the corresponding solution y of System (1.1) satisfies*

$$\Pi_{E_J^{-1}} y(t_0) = 0,$$

then we have the following dissipation estimate

$$\|y(t)\|_{H^{-1}(\Omega)^n} \leq Ce^{-\lambda_{J+1}^{\Omega_2}(t-t_0)} \|y(t_0)\|_{H^{-1}(\Omega)^n}, \quad \forall t \in (t_0, t_1).$$

[2] and [TT11, Theorem 1.5] when Ω_2 is a rectangular domain.

Proof. Let $y(t_0) = -\Delta \tilde{y}_0$, $\tilde{y}_0 \in H_0^1(\Omega)^n$. The assumption $\Pi_{E_J^{-1}} y(t_0) = 0$ translates into $\Pi_{E_J}\tilde{y}_0 = 0$.

Let $\tilde{y}$ be the solution in $H_0^1(\Omega)^n$ to

$$\begin{cases} \partial_t \tilde{y} &= \Delta \tilde{y} + A\tilde{y} \quad \text{in } (t_0,t_1)\times\Omega, \\ \tilde{y} &= 0 \quad \text{on } (t_0,t_1)\times\partial\Omega, \\ \tilde{y}(t_0) &= \tilde{y}_0 \quad \text{in } \Omega. \end{cases}$$

Since the matrix A is constant, we can check that

$$y = -\Delta\tilde{y} \quad \text{in } (t_0,t_1)\times\Omega,$$

and thus

$$\|y(t)\|_{H^{-1}(\Omega)^n} = \|\tilde{y}(t)\|_{H_0^1(\Omega)^n}, \quad \|y(t_0)\|_{H^{-1}(\Omega)^n} = \|\tilde{y}_0\|_{H_0^1(\Omega)^n}.$$

As a consequence it only remains to prove the dissipation for regular data, namely

$$\|\tilde{y}(t)\|_{H_0^1(\Omega)^n} \leq C e^{-\lambda_{J+1}^{\Omega_2}(t-t_0)} \|\tilde{y}_0\|_{H_0^1(\Omega)^n}, \quad \forall t\in(t_0,t_1),$$

for $\tilde{y}_0$ such that $\Pi_{E_J}\tilde{y}_0 = 0$ i.e. of the form (see Lemma 2.1)

$$\tilde{y}_0 = \sum_{j=J+1}^{+\infty} \tilde{y}_{0,j}\phi_j^{\Omega_2}, \quad \tilde{y}_{0,j} = \left\langle \tilde{y}_0, \phi_j^{\Omega_2}\right\rangle_{L^2(\Omega_2)^n} \in H_0^1(\Omega_1)^n.$$

Since $\Pi_{E_J}\tilde{y}_0 = 0$ and A is constant, we have $\Pi_{E_J}\tilde{y}(t) = 0$ for every $t\in(t_0,t_1)$ and as a result the following Poincaré inequality holds

$$\lambda_{J+1}^{\Omega_2}\|\tilde{y}(t)\|^2_{L^2(\Omega)^n} \leq \|\nabla\tilde{y}(t)\|^2_{L^2(\Omega)^n} \quad \forall t\in(t_0,t_1).$$

Combined to Young's inequality this leads to

$$\lambda_{J+1}^{\Omega_2}\|\nabla\tilde{y}(t)\|^2_{L^2(\Omega)^n} \leq 4\|\Delta\tilde{y}(t)\|^2_{L^2(\Omega)^n} \quad \text{for a.e. } t\in(t_0,t_1).$$

Using now standard energy estimates and the fact that the matrix A is constant and stable (see Remark 1), we finally obtain the desired dissipation

$$\|\tilde{y}(t)\|_{H_0^1(\Omega)^n} \leq C e^{-\lambda_{J+1}^{\Omega_2}(t-t_0)} \|\tilde{y}_0\|_{H_0^1(\Omega)^n}.$$

□

2.4. Lebeau-Robbiano time procedure. We are now ready to prove Theorem 1.3.

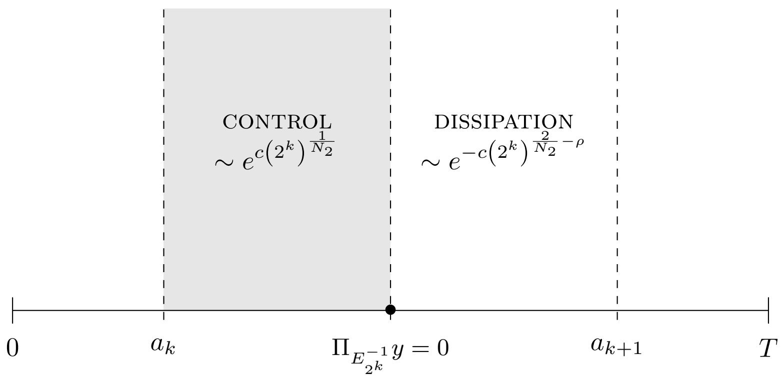
Let $y_0 \in H^{-1}(\Omega)^n$ be fixed. Let us decompose the interval $[0,T)$ as follows

$$[0,T) = \bigcup_{k=0}^{+\infty}[a_k, a_{k+1}],$$

with

$$a_0 = 0, \quad a_{k+1} = a_k + 2T_k, \quad T_k = M2^{-k\rho}.$$

where $\rho\in(0,\frac{1}{N_2})$ and $M = \frac{T}{2}(1-2^{-\rho})$ has been determined to ensure that $2\sum_{k=0}^{+\infty} T_k = T$.



We define the control v and the corresponding solution y piecewisely and by induction as follows

$$v(t) = \begin{cases} v\left(\Pi_{E_{2^k}^{-1}} y(a_k)\right)(t) & \text{if } t \in (a_k, a_k + T_k), \\ 0 & \text{if } t \in (a_k + T_k, a_{k+1}). \end{cases}$$

Let us show that v belongs to $L^2(0,T;L^2(\partial\Omega)^m)$ and steers y to 0 at time T.

Step 1 : Estimate on the interval $[a_k, a_k + T_k]$. From the continuous dependence with respect to the data (1.2) and since $T_k \leq T$ we know that

$$\|y(a_k + T_k)\|_{H^{-1}(\Omega)^n} \leq C\left(\|y(a_k)\|_{H^{-1}(\Omega)^n} + \|v\|_{L^2(a_k, a_k+T_k; L^2(\partial\Omega)^m)}\right). \tag{2.4}$$

Using the estimate of the cost of the control (2.2) we have

$$\|v\|_{L^2(a_k, a_k+T_k; L^2(\partial\Omega)^m)} \leq C C_{T_k}^{\Omega_1} e^{C\sqrt{\lambda_{2^k}^{\Omega_2}}} \left\|\Pi_{E_{2^k}^{-1}} y(a_k)\right\|_{H^{-1}(\Omega)^n}$$

and since $\left\|\Pi_{E_{2^k}^{-1}}\right\|_{\mathcal{L}(H^{-1})} \leq 1$, this gives

$$\|v\|_{L^2(a_k, a_k+T_k; L^2(\partial\Omega)^m)} \leq C C_{T_k}^{\Omega_1} e^{C\sqrt{\lambda_{2^k}^{\Omega_2}}} \|y(a_k)\|_{H^{-1}(\Omega)^n}.$$

Using now the estimate of $C_T^{\Omega_1}$ with respect to T (assumption (1.8)), this leads to

$$\|v\|_{L^2(a_k, a_k+T_k; L^2(\partial\Omega)^m)} \leq c e^{c\left(\frac{1}{T_k} + \sqrt{\lambda_{2^k}^{\Omega_2}}\right)} \|y(a_k)\|_{H^{-1}(\Omega)^n}.$$

On the other hand, Weyl's asymptotic formula states that

$$\sqrt{\lambda_{2^k}^{\Omega_2}} \underset{+\infty}{\sim} C\left(2^k\right)^{\frac{1}{N_2}}$$

and (by the choice of ρ)

$$\frac{1}{T_k} = \frac{1}{M} 2^{k\rho} \leq C 2^{\frac{k}{N_2}},$$

so that

$$\|v\|_{L^2(a_k, a_k+T_k; L^2(\partial\Omega)^m)} \leq C e^{C 2^{\frac{k}{N_2}}} \|y(a_k)\|_{H^{-1}(\Omega)^n}. \tag{2.5}$$

Combined to (2.4) this yields

$$\begin{aligned} \|y(a_k + T_k)\|_{H^{-1}(\Omega)^n} &\leq C\left(1 + e^{C 2^{\frac{k}{N_2}}}\right) \|y(a_k)\|_{H^{-1}(\Omega)^n} \\ &\leq C e^{C 2^{\frac{k}{N_2}}} \|y(a_k)\|_{H^{-1}(\Omega)^n}. \end{aligned} \tag{2.6}$$

Step 2 : Estimate on the interval $[a_k+T_k, a_{k+1}]$. Since $\Pi_{E_{2^k}^{-1}} y(a_k+T_k)=0$, the dissipation (Proposition 2.4) gives

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{-\lambda^{\Omega_2}_{2^k+1}T_k}\|y(a_k+T_k)\|_{H^{-1}(\Omega)^n}. \tag{2.7}$$

Step 3 : Final estimate. From (2.7) and (2.6) we deduce

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{-\lambda^{\Omega_2}_{2^k+1}T_k + C2^{\frac{k}{N_2}}}\|y(a_k)\|_{H^{-1}(\Omega)^n}.$$

By induction we obtain

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{\sum_{p=0}^k\left(-\lambda^{\Omega_2}_{2^p+1}T_p + C2^{\frac{p}{N_2}}\right)}\|y_0\|_{H^{-1}(\Omega)^n}.$$

Since

$$\lambda^{\Omega_2}_{2^p+1}T_p \underset{+\infty}{\sim} C(2^p+1)^{\frac{2}{N_2}}2^{-p\rho} \geq C'(2^p)^{\frac{2}{N_2}-\rho},$$

we obtain

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{\sum_{p=0}^k\left(-C'(2^p)^{\frac{2}{N_2}-\rho} + C(2^p)^{\frac{1}{N_2}}\right)}\|y_0\|_{H^{-1}(\Omega)^n}.$$

Since $\rho < \frac{1}{N_2}$, there exists a $p_0 \geq 1$ such that

$$-C'(2^p)^{\frac{2}{N_2}-\rho} + C(2^p)^{\frac{1}{N_2}} \leq -C''(2^p)^{\frac{2}{N_2}-\rho}, \quad \forall p \geq p_0. \tag{2.8}$$

It follows that, for $k \geq p_0$, we have

$$\sum_{p=0}^k\left(-C'(2^p)^{\frac{2}{N_2}-\rho} + C(2^p)^{\frac{1}{N_2}}\right) \leq C''' - C''\sum_{p=p_0}^k (2^p)^{\frac{2}{N_2}-\rho} \leq C''' - C''\left(2^k\right)^{\frac{2}{N_2}-\rho}.$$

So that, finally,

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{-C\left(2^k\right)^{\frac{2}{N_2}-\rho}}\|y_0\|_{H^{-1}(\Omega)^n}. \tag{2.9}$$

Step 4 : The function v is a control. Estimates (2.5) and (2.9) show that the function v is in $L^2(0,T;L^2(\partial\Omega))$:

$$\|v\|^2_{L^2(0,T;L^2(\partial\Omega)^m)} = \sum_{k=0}^{+\infty}\|v\|^2_{L^2(a_k,a_k+T_k;L^2(\partial\Omega)^m)} \leq C\underbrace{\left(\sum_{k=0}^{+\infty}e^{C2^{\frac{k}{N_2}}-C'\left(2^k\right)^{\frac{2}{N_2}-\rho}}\right)}_{<+\infty \text{ by } (2.8)}\|y_0\|^2_{H^{-1}(\Omega)^n}.$$

Moreover, estimate (2.9) also shows that the function v is indeed a control:

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \xrightarrow[k\to+\infty]{} 0 = \|y(T)\|_{H^{-1}(\Omega)^n}.$$

3. Cost of the one-dimensional boundary null-control. We prove here Theorem 1.4 assuming Theorem 1.5 is proved (see the next section). All along this part we shall use the notations of [AKBGBdT11a].

3.1. Arrangement and properties of the eigenvalues. Let us first recall that the Dirichlet eigenvalues of the Laplacian $-\partial^2_{xx}$ on $(0,\pi)$ (with domain $H^2(0,\pi)\cap H^1_0(0,\pi)$) are $\lambda_k = k^2$, $k \geq 1$.

We denote by $\{\mu_l\}_{l\in[\![1,p]\!]} \subset \mathbb{C}$ the set of distinct eigenvalues of A^*. For $l \in [\![1,p]\!]$, we denote the dimension of the eigenspace of A^* associated with μ_l by n_l and the size of its Jordan chains by $\tau_{l,j}$, $j \in [\![1,n_l]\!]$. In [AKBGBdT11a, Case 2, p. 583], it is shown that we can always assume that $\tau_{l,j} = \tau_l$ is independent of j. Finally, we set $\widehat{n} = \max_{l\in[\![1,p]\!]} n_l$.

We assume that the set $\{\mu_l\}_{l\in[\![1,p]\!]}$ is arranged in the following (non unique) way

$$\forall l \in [\![1,p-1]\!], \qquad \begin{cases} \Re(\mu_l) \geq \Re(\mu_{l+1}), \\ |\mu_l| \leq |\mu_{l+1}| \ \text{ if } \Re(\mu_l) = \Re(\mu_{l+1}). \end{cases} \tag{3.1}$$

We should point out that in [AKBGBdT11a, page 562], it is assumed that $\{\mu_l\}_{l\in[\![1,p]\!]}$ is ordered in such a way that $\widehat{n} = n_1$. Actually, this is only used for commodity and the same reasoning holds if we take $\widehat{n}$ instead of n_1.

Let us now recall that the eigenvalues of the operator $\partial^2_{xx} + A^*$ (with domain $H^2(0,\pi)^n \cap H^1_0(0,\pi)^n$) are given by $-\lambda_k + \mu_i$, $k \geq 1$ and $i \in [\![1,p]\!]$. Moreover, there exists $k_0 \geq 1$ such that

$$-\lambda_k + \mu_i \neq -\lambda_l + \mu_j, \tag{3.2}$$

for every $k \geq k_0$, $l \geq 1$, $l \neq k$, and $i,j \in [\![1,p]\!]$ with $i \neq j$ (see [AKBGBdT11a, Proposition 3.2]).

From (3.1), we see that there exists $k_1 \geq 1$ large enough so that

$$2\lambda_{k_1}\left(\Re(\mu_l) - \Re(\mu_{l+1})\right) + |\mu_{l+1}|^2 - |\mu_l|^2 \geq 0,$$

for every $l \in [\![1,p-1]\!]$. Therefore, we deduce that

$$|\lambda_k - \mu_l| \leq |\lambda_k - \mu_{l+1}|, \tag{3.3}$$

for every $k \geq k_1$ and $l \in [\![1,p-1]\!]$.

Finally, let $k_2 \geq 1$ be large enough so that

$$1 + |\lambda_k - \mu_i| \leq |\lambda_{k+1} - \mu_j|, \tag{3.4}$$

for every $k \geq k_2$ and $i,j \in [\![1,p]\!]$ with $i \neq j$, which is always possible since $\lambda_k = k^2$.

We set

$$K_0 = \max\{k_0, k_1, k_2\}.$$

To this K_0 we associate $\widetilde{p} \geq 1$, the number of distinct eigenvalues of the matrix $A^*_{K_0}$ defined in (1.6). Let $\{\gamma_\ell\}_{\ell\in[\![1,\widetilde{p}]\!]} \subset \{-\lambda_k + \mu_l\}_{k\in[\![1,K_0]\!], l\in[\![1,p]\!]}$ be the set of distinct eigenvalues of $A^*_{K_0}$ arranged in such a way that $|\gamma_\ell| \leq |\gamma_{\ell+1}|$ for every $\ell \in [\![1,\widetilde{p}-1]\!]$.

For $\ell \in [\![1,\widetilde{p}]\!]$, the dimension of the eigenspace of $A^*_{K_0}$ associated with γ_ℓ is denoted by N_ℓ, and the size of its Jordan chains by $\widetilde{\tau}_{\ell,j}$, $j \in [\![1,N_\ell]\!]$. Since we assumed that $\tau_{l,j} = \tau_l$ it follows that $\widetilde{\tau}_{\ell,j} = \widetilde{\tau}_\ell$ is also independent of j. Finally, we set $\widehat{N} = \max_{\ell\in[\![1,\widetilde{p}]\!]} N_\ell$.

We choose to arrange the eigenvalues $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ of the operator $-(\Delta + A^*)$ as follows:

$$\begin{cases} \Lambda_\ell = -\gamma_\ell, \quad \text{for } \ell \in [\![1,\widetilde{p}]\!], \\ \Lambda_{\widetilde{p}+i} = \lambda_{K_0+j} - \mu_l, \quad \text{with} \quad j = \left\lfloor \frac{i-1}{p} \right\rfloor + 1 \text{ and } l = i - \left\lfloor \frac{i-1}{p} \right\rfloor p, \quad \text{for } i \geq 1. \end{cases}$$

Observe that the sequence $\{\Lambda_k\}_{k\geq 1}$ satisfies the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ of Theorem 1.5:

- $(\mathcal{H}_1)$ follows from (3.2).
- $(\mathcal{H}_2)$ holds because the matrix A is stable (see Remark 1).
- $(\mathcal{H}_3)$ is clear since $|\Im(\Lambda_k)| \leq \max_{l\in[\![1,p]\!]} |\Im(\mu_l)|$ and $\Re(\Lambda_k) \geq \lambda_1 - \max_{l\in[\![1,p]\!]} \Re(\mu_l)$ (which is positive since A^* is stable).

- $(\mathcal{H}_4)$ is a consequence of (3.3) and (3.4).
- Finally, let us show that $(\mathcal{H}_5)$ holds for q large enough. Let $k = \widetilde{p} + i_k$ and $n = \widetilde{p} + i_n$ (the case $k \leq \widetilde{p}$ or $n \leq \widetilde{p}$ is simpler). Let j_k, j_n and l_k, l_n be such that $\Lambda_k = \lambda_{K_0+j_k} - \mu_{l_k}$ and $\Lambda_n = \lambda_{K_0+j_n} - \mu_{l_n}$. We have

$$\begin{aligned}|\Lambda_n - \Lambda_k|^2 &= |\lambda_{K_0+j_k} - \lambda_{K_0+j_n} + \mu_{l_n} - \mu_{l_k}|^2 \geq \Big| |\lambda_{K_0+j_k} - \lambda_{K_0+j_n}| - |\mu_{l_n} - \mu_{l_k}| \Big|^2 \\ &\geq |\lambda_{K_0+j_k} - \lambda_{K_0+j_n}|^2 - 2|\lambda_{K_0+j_k} - \lambda_{K_0+j_n}|\,|\mu_{l_n} - \mu_{l_k}| + |\mu_{l_n} - \mu_{l_k}|^2.\end{aligned}$$

Let us denote $m = \min_{\substack{1\leq l,l'\leq p\\ l\neq l'}} |\mu_l - \mu_{l'}|$, $M = \max_{\substack{1\leq l,l'\leq p\\ l\neq l'}} |\mu_l - \mu_{l'}|$, $d = |j_k - j_n|$, $s = j_k + j_n$ and $x = d(s + 2K_0)$. Thus,

$$|\Lambda_n - \Lambda_k|^2 \geq x^2 - 2Mx + m.$$

On the other hand, since $|i_k - i_n| < p(|j_k - j_n| + 1)$ and $i_k + i_n \leq p(j_k + j_n) + 2$, we have

$$\left|k^2 - n^2\right|^2 = |i_k - i_n|^2 (i_k + i_n + 2\widetilde{p})^2 \leq p^2(d+1)^2(sp + 2 + 2\widetilde{p})^2$$

By assumption $d, s \to +\infty$, so that

$$\left|k^2 - n^2\right|^2 \leq Cd^2(s + 2K_0)^2 = Cx^2.$$

Taking for instance $\rho = 1/\sqrt{2C}$ and x large enough we obtain the first property of $(\mathcal{H}_5)$. The second property is actually satisfied for any q.

The counting function. We recall that the counting function $\mathcal{N}$ associated with the sequence $\{\Lambda_k\}_{k\geq 1}$ is given by

$$\mathcal{N}(r) = \#\{k : |\Lambda_k| \leq r\}, \quad \forall r > 0.$$

This function $\mathcal{N}$ is piecewise constant and non-decreasing on the interval $[0, +\infty)$. Thanks to $(\mathcal{H}_5)$ we have $\lim_{k\to+\infty} |\Lambda_k| = +\infty$, so that $\mathcal{N}(r) < +\infty$ for every $r \in [0, +\infty)$ and $\lim_{r\to+\infty} \mathcal{N}(r) = +\infty$. Moreover, $(\mathcal{H}_4)$ shows that, for every $r > 0$, we have

$$\mathcal{N}(r) = n \Longleftrightarrow (|\Lambda_n| \leq r \text{ and } |\Lambda_{n+1}| > r), \tag{3.5}$$

so that, in particular, we have

$$\sqrt{|\Lambda_{\mathcal{N}(r)}|} \leq \sqrt{r} < \sqrt{|\Lambda_{\mathcal{N}(r)+1}|}.$$

On the other hand, from the very definition of Λ_k for $k > \widetilde{p}$, we have

$$\left(\frac{\mathcal{N}(r)}{p} + \widetilde{K_0}\right)^2 - M \leq \left|\Lambda_{\mathcal{N}(r)}\right| \leq \left(\frac{\mathcal{N}(r)}{p} + \widetilde{\widetilde{K_0}}\right)^2 + M, \quad \text{for any } r \text{ s.t. } \mathcal{N}(r) > \widetilde{p},$$

where $M = \max_{l\in[\![1,p]\!]} |\mu_l|$, $\widetilde{K_0} = K_0 - \frac{\widetilde{p}+1}{p} + 1$ and $\widetilde{\widetilde{K_0}} = \widetilde{K_0} + 1$. Combining the two previous estimates, it is not difficult to obtain the last assumption $(\mathcal{H}_6)$ of Theorem 1.5.

3.2. The moment problem. In [AKBGBdT11a] it has been proved (Proposition 5.1) that, under the assumption (1.5), System (1.9) is null-controllable at time T if for every $q \in [\![1, \widehat{N}]\!]$ there exists a solution $u_q \in L^2(0, T)$ to the moments problem

$$\left\{\begin{aligned} &\int_0^T \frac{t^\nu}{\nu!} e^{\overline{\gamma_\ell} t}\, u_q(t)\,dt = c_{\ell,\nu,q}(y_0; T), \quad \forall \ell \in [\![1, \widetilde{p}]\!], \forall \nu \in [\![0, \widetilde{\tau}_\ell - 1]\!], \\ &\int_0^T \frac{t^\sigma}{\sigma!} e^{(-\lambda_k + \overline{\mu_l})t}\, u_q(t)\,dt = d^k_{l,\sigma,q}(y_0; T), \quad \forall k > K_0, \forall l \in [\![1, p]\!], \forall \sigma \in [\![0, \tau_l - 1]\!], \end{aligned}\right. \tag{3.6}$$

where $c_{\ell,\nu,q}$ and $d^k_{l,\sigma,q}$ are given in [AKBGBdT11a, Proposition 5.1]. The precise definition of those terms is not really important here, however we recall that they satisfy the following estimates (see [AKBGBdT11a, Equations (49) and (52)])

$$\begin{aligned} |c_{\ell,\nu,q}(y_0;T)| &\leq C\left\|e^{A^*_{K_0}T}\right\|_{\mathcal{M}_{nK_0}(\mathbb{C})}\|y_0\|_{H^{-1}(0,\pi)^n} \\ &\leq Ce^{CT}\|y_0\|_{H^{-1}(0,\pi)^n}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \left|d^k_{l,\sigma,q}(y_0;T)\right| &\leq \frac{C}{k}\left\|e^{(-\lambda_k+A^*)T}\right\|_{\mathcal{M}_n(\mathbb{C})}\left|\langle y_0,\phi_k\rangle_{H^{-1},H^1_0(0,\pi)}\right|_{\mathbb{C}^n} \\ &\leq Ce^{CT}\frac{\sqrt{\lambda_k}}{k}e^{-\lambda_k T}\|y_0\|_{H^{-1}(0,\pi)^n}. \end{aligned} \tag{3.8}$$

The control $v(t)$ is then given as a linear combination of $u_q(T-t)$, $q\in[\![1,\widehat{N}]\!]$, and as a result satisfies

$$\|v\|_{L^2(0,T)^m} \leq C\max_{q\in[\![1,\widehat{N}]\!]}\|u_q\|_{L^2(0,T)}. \tag{3.9}$$

Assume for the moment that Theorem 1.5 is proved. Let $T_0>0$ be the time given by Theorem 1.5 and set

$$\eta = \max\left\{\tau_l, \widetilde{\tau}_\ell, \mid l\in[\![1,p]\!],\ \ell\in[\![1,\widetilde{p}]\!]\right\}.$$

For $T<T_0$ we can then introduce the biorthogonal family $\{\varphi_{k,j}\}_{k\geq 1, j\in[\![0,\eta-1]\!]}\subset L^2(-T/2,T/2)$ associated with the sequence $\{\Lambda_k\}_{k\geq 1}$. As we need to work on the interval $(-T/2,T/2)$, we perform the change of variable $s=t-\frac{T}{2}$ in (3.6) and obtain

$$\begin{cases} \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1}{\nu!}\left(s+\frac{T}{2}\right)^\nu e^{\overline{\gamma_\ell}s}\,u_q\left(s+\frac{T}{2}\right)ds = e^{-\frac{T}{2}\overline{\gamma_\ell}}c_{\ell,\nu,q}(y_0;T), \quad \forall\ell\in[\![1,\widetilde{p}]\!],\forall\nu\in[\![0,\widetilde{\tau}_\ell-1]\!], \\ \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}}\frac{1}{\sigma!}\left(s+\frac{T}{2}\right)^\sigma e^{(-\lambda_k+\overline{\mu_l})s}\,u_q\left(s+\frac{T}{2}\right)ds = e^{-(-\lambda_k+\overline{\mu_l})\frac{T}{2}}d^k_{l,\sigma,q}(y_0;T), \quad \forall k>K_0,\forall l\in[\![1,p]\!],\forall\sigma\in[\![0,\tau_l-1]\!]. \end{cases}$$

Using the binomial formula $\left(s+\frac{T}{2}\right)^J=\sum_{j=0}^J\binom{J}{j}s^{J-j}\left(\frac{T}{2}\right)^j$ we finally have

$$\begin{cases} \displaystyle\sum_{j=0}^{\nu}\binom{\nu}{j}\left(\frac{T}{2}\right)^j\int_{-\frac{T}{2}}^{\frac{T}{2}}s^{\nu-j}e^{\overline{\gamma_\ell}s}\,u_q\left(s+\frac{T}{2}\right)ds = \widehat{c_{\ell,\nu,q}}(y_0;T), \quad \forall\ell\in[\![1,\widetilde{p}]\!],\forall\nu\in[\![0,\widetilde{\tau}_\ell-1]\!], \\ \displaystyle\sum_{j=0}^{\sigma}\binom{\sigma}{j}\left(\frac{T}{2}\right)^j\int_{-\frac{T}{2}}^{\frac{T}{2}}s^{\sigma-j}e^{(-\lambda_k+\overline{\mu_l})s}\,u_q\left(s+\frac{T}{2}\right)ds = \widehat{d^k_{l,\sigma,q}}(y_0;T), \quad \forall k>K_0,\forall l\in[\![1,p]\!],\forall\sigma\in[\![0,\tau_l-1]\!], \end{cases}$$

with

$$\widehat{c_{\ell,\nu,q}}(y_0;T)=\nu!e^{-\frac{T}{2}\overline{\gamma_\ell}}c_{\ell,\nu,q}(y_0;T), \quad \widehat{d^k_{l,\sigma,q}}(y_0;T)=\sigma!e^{-(-\lambda_k+\overline{\mu_l})\frac{T}{2}}d^k_{l,\sigma,q}(y_0;T). \tag{3.10}$$

For $T<T_0$, a solution to the moments problem (3.6) is then given for every $t\in(0,T)$ by (note that $-\lambda_k+\mu_l=\Lambda_{\widetilde{p}+(k-K_0-1)p+l}$ for $k>K_0$)

$$\begin{aligned} u_q(t) = \sum_{\ell=1}^{\widetilde{p}}\sum_{\nu=0}^{\widetilde{\tau}_\ell-1}\widehat{\widehat{c_{\ell,\nu,q}}}(y_0;T)\varphi_{\ell,\nu}\left(t-\frac{T}{2}\right) \\ + \sum_{k>K_0}\sum_{l=1}^{p}\sum_{\sigma=0}^{\tau_l-1}\widehat{\widehat{d^k_{l,\sigma,q}}}(y_0;T)\varphi_{\widetilde{p}+(k-K_0-1)p+l,\sigma}\left(t-\frac{T}{2}\right), \end{aligned}$$

provided that u_q lies in $L^2(0,T)$ (see below), and where $\widehat{\widehat{c_{\ell,\nu,q}}}$ and $\widehat{\widehat{d^k_{l,\sigma,q}}}$ solve the triangular systems

$$P(T)\begin{pmatrix}\widehat{\widehat{c_{\ell,0,q}}}\\ \vdots\\ \widehat{\widehat{c_{\ell,\widetilde{\tau}_\ell-1,q}}}\end{pmatrix}=\begin{pmatrix}\widehat{c_{\ell,0,q}}\\ \vdots\\ \widehat{c_{\ell,\nu,q}}\end{pmatrix},\quad Q(T)\begin{pmatrix}\widehat{\widehat{d^k_{l,0,q}}}\\ \vdots\\ \widehat{\widehat{d^k_{l,\tau_l-1,q}}}\end{pmatrix}=\begin{pmatrix}\widehat{d^k_{l,0,q}}\\ \vdots\\ \widehat{d^k_{l,\tau_l-1,q}}\end{pmatrix},$$

where the coefficients of $P(T)$ and $Q(T)$ are respectively given for $i\geq j$ by $p_{ij}(T)=\binom{i-1}{j-1}\left(\frac{T}{2}\right)^{i-j}$, $q_{ij}(T)=\binom{i-1}{j-1}\left(\frac{T}{2}\right)^{i-j}$ and $p_{ij}(T)=q_{ij}(T)=0$ otherwise. Observe that

$$\left\|P(T)^{-1}\right\|_{\mathcal{M}_{\widetilde{\tau}_\ell-1}(\mathbb{C})}\leq CT^{\widetilde{\tau}_\ell-1},\quad \left\|Q(T)^{-1}\right\|_{\mathcal{M}_{\tau_l-1}(\mathbb{C})}\leq CT^{\tau_l-1}.$$

From this, the definition (3.10) of $\widehat{c_{\ell,\nu,q}}$ and $\widehat{d^k_{l,\sigma,q}}$, and the estimates (3.7) and (3.8) of $c_{\ell,\nu,q}$ and $d^k_{l,\sigma,q}$, we obtain

$$\left|\widehat{\widehat{c_{\ell,\nu,q}}}(y_0;T)\right|\leq CT^{\widetilde{\tau}_\ell-1}\left|e^{-\frac{T}{2}\overline{\gamma_\ell}}\right|e^{CT}\|y_0\|_{H^{-1}(0,\pi)^n}\leq Ce^{CT}\|y_0\|_{H^{-1}(0,\pi)^n},\tag{3.11}$$

and

$$\begin{aligned}\left|\widehat{\widehat{d^k_{l,\sigma,q}}}(y_0;T)\right|&\leq CT^{\tau_l-1}\left|e^{-(-\lambda_k+\overline{\mu_l})\frac{T}{2}}\right|\frac{\sqrt{\lambda_k}}{k}e^{CT}e^{-\lambda_kT}\|y_0\|_{H^{-1}(0,\pi)^n},\\ &\leq Ce^{CT}\frac{\sqrt{\lambda_k}}{k}e^{-\lambda_k\frac{T}{2}}\|y_0\|_{H^{-1}(0,\pi)^n}.\end{aligned}\tag{3.12}$$

It remains to prove that $u_q\in L^2(0,T)$ and to estimate its norm with respect to T and y_0. This is actually thanks to the estimate (1.12) that this latter can be achieved. Indeed, using also (3.11) and (3.12) we have

$$\begin{aligned}\|u_q\|_{L^2(0,T)}&\leq Ce^{CT}\sum_{\ell=1}^{\widetilde{p}}e^{C\sqrt{-\Re(\gamma_\ell)}+\frac{C}{T}}\|y_0\|_{H^{-1}(0,\pi)^n},\\ &\quad+Ce^{CT}\sum_{k>K_0}\frac{\sqrt{\lambda_k}}{k}e^{-\lambda_k\frac{T}{2}}\sum_{l=1}^{p}e^{C\sqrt{\lambda_k-\Re(\mu_l)}+\frac{C}{T}}\|y_0\|_{H^{-1}(0,\pi)^n},\\ &\leq Ce^{CT+\frac{C}{T}}\left(1+\sum_{k>K_0}\frac{\sqrt{\lambda_k}}{k}e^{-\lambda_k\frac{T}{2}+C\sqrt{\lambda_k}}\right)\|y_0\|_{H^{-1}(0,\pi)^n}.\end{aligned}\tag{3.13}$$

Let us now estimate the series. Young's inequality gives

$$C\sqrt{\lambda_k}\leq\lambda_k\frac{T}{4}+\frac{C^2}{T},$$

for every $k\geq 1$ and $T>0$, so that

$$-\lambda_k\frac{T}{2}+C\sqrt{\lambda_k}\leq-\lambda_k\frac{T}{4}+\frac{C^2}{T}.$$

Thus, using also that $\lambda_k=k^2$, we obtain

$$\sum_{k>K_0}\frac{\sqrt{\lambda_k}}{k}e^{-\lambda_k\frac{T}{2}+C\sqrt{\lambda_k}}\leq e^{\frac{C}{T}}\sum_{k\geq 0}e^{-k^2\frac{T}{4}}.$$

A comparison with the Gauss integral gives

$$\sum_{k\geq 0}e^{-k^2\frac{T}{4}}\leq 2\sqrt{\frac{4\pi}{T}}\leq Ce^{\frac{C}{T}}.$$

Coming back to (3.13) we then have

$$\|u_q\|_{L^2(0,T)} \leq Ce^{CT+\frac{C}{T}} \|y_0\|_{H^{-1}(0,\pi)^n}.$$

Finally, (3.9) gives, for every $T < T_0$,

$$\|v\|_{L^2(0,T)} \leq Ce^{\frac{C}{T}} \|y_0\|_{H^{-1}(0,\pi)^n}.$$

Thus, when $T < T_0$ we have obtained a null-control to System (1.9) which satisfies the desired estimate. The case $T \geq T_0$ is actually reduced to the previous one. Indeed, any continuation by zero of a control on $(0, T_0/2)$ is a control on $(0,T)$ and the estimate follows from the decrease of the cost with respect to the time.

4. Biorthogonal families to complex matrix exponentials.. This section is devoted to the proof of Theorem 1.5.

4.1. Idea of the proof. For any $\eta \geq 1$ and T small enough (depending on η), we have to construct a family $\{\varphi_{k,j}\}_{k\geq 1, j\in [\![0,\eta-1]\!]}$ in $L^2(-T/2, T/2)$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi_{k,j}(t) t^\nu e^{-\overline{\Lambda_l} t}\, dt = \delta_{kl}\delta_{j\nu},$$

for every $k, l \geq 1$ and $j, \nu \in [\![0, \eta-1]\!]$, with in addition the following bound

$$\|\varphi_{k,j}\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \leq Ce^{C\sqrt{\Re(\Lambda_k)}+\frac{C}{T}},$$

for any $k \geq 1$ and $j \in [\![0, \eta-1]\!]$.

The idea is to use the Fourier transform with the help of the Paley-Wiener theorem (see [Rud74, Theorem 19.3]) that we recall here.

THEOREM 4.1. *Let Φ be an entire function of exponential type $T/2$ (that is $|\Phi(z)| \leq Ce^{\frac{T}{2}|z|}$ for all $z \in \mathbb{C}$[3]) such that*

$$\|\Phi\|^2_{L^2(-\infty,+\infty)} = \int_{-\infty}^{+\infty} |\Phi(x)|^2\, dx < +\infty.$$

Then, there exists $\varphi \in L^2(-T/2, T/2)$ such that

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi(t) e^{itz}\, dt, \quad \forall z \in \mathbb{C}. \tag{4.1}$$

Moreover, the Plancherel theorem gives

$$\|\varphi\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} = \|\Phi\|_{L^2(-\infty,+\infty)}.$$

Observe that the function in (4.1) is infinitely derivable on $\mathbb{C}$ with, for every $\nu \in [\![0, \eta-1]\!]$,

$$\Phi^{(\nu)}(z) = \frac{i^\nu}{\sqrt{2\pi}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi(t) t^\nu e^{itz}\, dt, \quad \forall z \in \mathbb{C}.$$

Thus, Theorem 1.5 will be proved if we manage to build suitable entire functions as stated in the following result.

THEOREM 4.2. *Assume that the sequence $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfies the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_6)$.*

[3]Here and only here, C may even depend on T without affecting the result.

There exists $T_0 > 0$ such that, for any $\eta \geq 1$ and $0 < T < T_0$, there exists a family $\{\Phi_{k,j}\}_{k\geq 1, j\in[\![0,\eta-1]\!]}$ of entire functions of exponential type $T/2$ satisfying

$$\Phi_{k,j}^{(\nu)}(i\overline{\Lambda_l}) = \frac{i^\nu}{\sqrt{2\pi}}\delta_{kl}\delta_{j\nu}, \quad \forall k, l \geq 1, \quad \forall j, \nu \in [\![0, \eta-1]\!], \tag{4.2}$$

and

$$\|\Phi_{k,j}\|_{L^2(-\infty,+\infty)} \leq Ce^{C\sqrt{\Re(\Lambda_k)}+\frac{C}{T}}, \tag{4.3}$$

for any $k \geq 1$ and $j \in [\![0, \eta-1]\!]$.

Remark 3. *A sequence $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfies the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_6)$ if and only if so does the sequence $\left\{\overline{\Lambda_k}\right\}_{k\geq 1}$. For this reason, and commodity, we will prove Theorem 4.2 for the sequence $\left\{\overline{\Lambda_k}\right\}_{k\geq 1}$.*

4.2. Proof of Theorem 4.2.

Some preliminary remarks. It is interesting to point out some properties of the sequence $\{\Lambda_k\}_{k\geq 1}$ which can be deduced from assumptions $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and $(\mathcal{H}_6)$.

1. First, under assumptions $(\mathcal{H}_4)$ and $(\mathcal{H}_6)$ we have that

$$\sum_{k\geq 1}\frac{1}{|\Lambda_k|} < +\infty. \tag{4.4}$$

   Indeed, using that $\mathcal{N}$ is piecewise constant and non-decreasing on the interval $[0, +\infty)$, we can write

$$\sum_{k\geq 1}\frac{1}{|\Lambda_k|} = \int_{|\Lambda_1|^-}^{+\infty}\frac{1}{r}\,d\mathcal{N}(r) = \int_{|\Lambda_1|}^{+\infty}\frac{1}{r^2}\mathcal{N}(r)\,dr \leq \int_{|\Lambda_1|}^{+\infty}\frac{\alpha+p\sqrt{r}}{r^2}\,dr = \frac{\alpha}{|\Lambda_1|}+\frac{2p}{\sqrt{|\Lambda_1|}} < +\infty.$$

2. Then, from assumption $(\mathcal{H}_3)$ we can also deduce the following behavior of the sequence $\{\Lambda_k\}_{k\geq 1}$

$$|\Lambda_k| - \Re(\Lambda_k) \leq \beta\sqrt{\Re(\Lambda_k)} \quad \text{and} \quad |\Lambda_k| \leq C\Re(\Lambda_k), \quad \forall k \geq 1. \tag{4.5}$$

   Indeed, one has

$$|\Lambda_k|^2 = \Re(\Lambda_k)^2 + \Im(\Lambda_k)^2 \leq \Re(\Lambda_k)^2 + \beta^2\Re(\Lambda_k) \leq \left(\Re(\Lambda_k) + \beta\sqrt{\Re(\Lambda_k)}\right)^2.$$

Let us now introduce the complex functions given, for every $z \in \mathbb{C}$, by

$$f(z) = \prod_{k\geq 1}\left(1-\frac{z}{\Lambda_k}\right), \quad f_n(z) = \prod_{\substack{k\geq 1\\ k\neq n}}\left(1-\frac{z}{\Lambda_k}\right). \tag{4.6}$$

Thanks to (4.4), the previous products are uniformly convergent on compact sets of $\mathbb{C}$ and therefore f and f_n are entire functions. Moreover, the zeros of f and f_n are exactly $\{\Lambda_k\}_{k\geq 1}$ and $\{\Lambda_k\}_{k\neq n}$ and they are zeros of multiplicity 1 (recall that the Λ_k are distinct by $(\mathcal{H}_1)$). For a proof of these facts we refer to [Rud74, Theorem 15.4].

On the other hand, let us fix $d = p\pi + 2$. For any $\tau > 0$ such that $\tau < d^2/2$ we define the real positive sequence $\{a_n\}_{n\geq 0}$ given by

$$a_n = \frac{d^2}{\tau^2} + \frac{4\left(n^2-1\right)}{d^2}, \quad \forall n \geq 0, \tag{4.7}$$

To this sequence we associate a complex function M defined by

$$M(z) = \prod_{n\geq 1}\frac{\sin(z/a_n)}{z/a_n}, \quad \forall z \in \mathbb{C}. \tag{4.8}$$

Since

$$\left|\frac{\sin(z)}{z}\right| \leq e^{|z|}, \quad \forall z \in \mathbb{C},$$

and $a_n \underset{+\infty}{\sim} Cn^2$, the previous product is uniformly convergent on compact sets of $\mathbb{C}$ and M is an entire function of exponential type $\tau_M > 0$, where

$$\tau_M = \sum_{n\geq 1} \frac{1}{a_n} < +\infty. \tag{4.9}$$

More precisely, M satisfies

$$|M(z)| \leq e^{\tau_M |z|}, \quad \forall z \in \mathbb{C}. \tag{4.10}$$

Observe that there is no constant in front of the term $e^{\tau_M|z|}$. This point will be very important in the sequel (see the proof of Proposition 4.3 in Appendix) to obtain estimates with constants C that do not depend on τ (which will play the role of T, see below). Note also that M has only real zeros since $\{a_n\}_{n\geq 1}$ is a real sequence. Finally, we will often use that $\tau_M < \tau$. This fact is proved in Lemma A.2 in Appendix.

Proof of Theorem 4.2. We follow some techniques developed in [AKBGBdT11a] (see in particular Lemma 4.4 in this reference).

Set $T_0 = d^2$ and, for any $0 < T < T_0$, set $\tau = \frac{T}{2\eta}$, in such a way that the condition $\tau < d^2/2$ holds. The function M defined above will then correspond to this value of τ.

Let us consider the functions

$$\begin{cases} \Phi_k(z) = \dfrac{1}{\eta!}\left[W_k(z)\right]^\eta, \quad W_k(z) = \dfrac{f(-iz)}{-if'(\Lambda_k)}\dfrac{M(z+\Im(\Lambda_k))}{M(i\Re(\Lambda_k))}, \\ \widetilde{\Phi}_k(z) = \dfrac{1}{\eta!}\left[\widetilde{W}_k(z)\right]^\eta \quad \widetilde{W}_k(z) = \dfrac{f_k(-iz)}{-if'(\Lambda_k)}\dfrac{M(z+\Im(\Lambda_k))}{M(i\Re(\Lambda_k))}, \end{cases} \tag{4.11}$$

defined for every $z \in \mathbb{C}$ and $k \geq 1$.

Let us already give some estimates for the functions W_k, $\widetilde{W}_k$ (and as result also for Φ_k and $\widetilde{\Phi}_k$) that will be used later:

Proposition 4.3. *Assume that the sequence $\{\Lambda_k\}_{k\geq 1}$ satisfies the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_6)$, and let $\tau < d^2/2$. Then, for any $k \geq 1$ and $z \in \mathbb{C}$,*

$$|W_k(z)| + |\widetilde{W}_k(z)| \leq e^{C\sqrt{|z|}+\tau_M(|z|-\Re(\Lambda_k))+C\sqrt{\Re(\Lambda_k)}+\frac{C}{\tau}}. \tag{4.12}$$

On the other hand, for any $k \geq 1$ and $x \in \mathbb{R}$,

$$|W_k(x)| + |\widetilde{W}_k(x)| \leq e^{-\sqrt{|x|}+C\sqrt{\Re(\Lambda_k)}+\frac{C}{\tau}}. \tag{4.13}$$

The proof of this rather technical proposition is given in Appendix. For now, let us continue with the proof of Theorem 4.2.

Since the function M only has real zeros, all the functions introduced in (4.11) are well-defined and they are entire functions. For every $l \geq 1$, $i\Lambda_l$ is a simple zero of the function W_k since Λ_l is a simple zero of f and $i\Lambda_l + \Im(\Lambda_k)$ is not a zero of M ($\Im[i\Lambda_l + \Im(\Lambda_k)] = \Re(\Lambda_l) \neq 0$ by $(\mathcal{H}_2)$). Thus, we deduce that, for every $l \geq 1$, $i\Lambda_l$ is a zero of Φ_k with exact multiplicity η, i.e.,

$$\Phi_k^{(\eta)}(i\Lambda_l) = [W_k'(i\Lambda_l)]^\eta \neq 0 \quad \text{and} \quad \Phi_k^{(\nu)}(i\Lambda_l) = 0, \quad \forall k, l \geq 1, \quad \forall \nu \in [\![0, \eta-1]\!].$$

Observe that, in particular $\Phi_k^{(\eta)}(i\Lambda_k) = 1$. At this point, the function $\Phi_{k,j} = \Phi_k$ then satisfies (4.2) for $l \neq k$.

For any $k \geq 1$, $j \in [\![0, \eta - 1]\!]$ and $z \in \mathbb{C}$, let us now set

$$f_{k,j}(z) = \frac{\Phi_k(z)}{(z - i\Lambda_k)^{\eta - j}} = \left(\frac{-1}{i\Lambda_k}\right)^{\eta} \widetilde{\Phi}_k(z)(z - i\Lambda_k)^j.$$

Note that, for $x \in \mathbb{R}$, we deduce from (4.13), $(\mathcal{H}_4)$ and (4.5), that

$$|f_{k,j}(x)| \leq Ce^{-\frac{\eta}{2}\sqrt{|x|} + C\sqrt{\Re(\Lambda_k)} + \frac{C}{\tau}}. \tag{4.14}$$

From the properties of the function Φ_k, we get

$$\begin{cases} f_{k,j}^{(\nu)}(i\Lambda_l) = 0, & \forall l \geq 1 \text{ with } l \neq k,\ \forall \nu \in [\![0, \eta - 1]\!], \\ f_{k,j}^{(\nu)}(i\Lambda_k) = 0, & \forall \nu \in [\![0, j - 1]\!], \\ f_{k,j}^{(j+r)}(i\Lambda_k) = \dfrac{(j+r)!}{(\eta + r)!}\Phi_k^{(\eta + r)}(i\Lambda_k), & \forall r \geq 0. \end{cases} \tag{4.15}$$

We look now for $\Phi_{k,j}$ in the following form

$$\Phi_{k,j}(z) = p(z) f_{k,j}(z),$$

with p a polynomial function of degree $\eta - j - 1$ which depends on k, j (for simplicity, this dependance is omitted in the notation).

As a consequence of inequality (4.12) and the fact that $\tau_M < \tau$, the function $\Phi_{k,j}$ is an entire function of exponential type $\eta\tau = T/2$.[4]

In view of (4.15), if we simply take $p = 1$, then the relations (4.2) are satisfied for $l \neq k$ and $l = k$ if $\nu < j$. Thus, in order to get (4.2), we have to choose p such that $\Phi_{k,j}^{(j)}(i\Lambda_k) = \frac{i^j}{\sqrt{2\pi}}$ and $\Phi_{k,j}^{(j+r)}(i\Lambda_k) = 0$ for $r \in [\![1, \eta - j - 1]\!]$, that is

$$\begin{cases} p(i\Lambda_k) = \dfrac{i^j}{\sqrt{2\pi}} \dfrac{1}{f_{k,j}^{(j)}(i\Lambda_k)} = \dfrac{i^j}{\sqrt{2\pi}} \dfrac{\eta!}{j!}, \\ \displaystyle\sum_{\ell=0}^{r-1} a_{r\ell}\, p^{(\ell)}(i\Lambda_k) + p^{(r)}(i\Lambda_k) = 0, \quad \forall r \in [\![1, \eta - j - 1]\!], \end{cases} \tag{4.16}$$

where

$$a_{r\ell} = \frac{\binom{j+r}{l}}{\binom{j+r}{r}} \frac{f_{k,j}^{(j+r-\ell)}(i\Lambda_k)}{f_{k,j}^{(j)}(i\Lambda_k)} = \frac{r!\eta!}{\ell!(\eta + r - \ell)!}\, \Phi_k^{(\eta + r - \ell)}(i\Lambda_k), \tag{4.17}$$

for every $r \in [\![1, \eta - j - 1]\!]$ and $\ell \in [\![0, r - 1]\!]$ (they are well-defined since $f_{k,j}^{(j)}(i\Lambda_k) \neq 0$).

These relations allow us to compute $p^{(r)}(i\Lambda_k)$ for every $r \in [\![0, \eta - j - 1]\!]$ and thus completely determine p which is then given by

$$p(z) = \sum_{r=0}^{\eta - j - 1} \frac{p^{(r)}(i\Lambda_k)}{r!}(z - i\Lambda_k)^r.$$

In order to get the bound (4.3) for $\Phi_{k,j}$, let us prove some estimates of the polynomial p previously constructed. If we set $P = \left(p^{(r)}(i\Lambda_k)\right)_{r \in [\![0, \eta - j - 1]\!]} \in \mathbb{C}^{\eta - j}$, then we can rewrite the

[4] the constant C such that $\left|\Phi_{k,j}(z)\right| \leq Ce^{\eta\tau|z|}$ for every $z \in \mathbb{C}$ depends on k, j, τ, etc... but this is not important as mentioned earlier.

identities in (4.16) as a linear system of the form $\mathbf{A}P = \mathbf{B}$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ a_{10} & 1 & \ddots & & \vdots \\ a_{20} & a_{21} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{\eta-j-1,0} & a_{\eta-j-1,1} & \cdots & a_{\eta-j-1,\eta-j-2} & 1 \end{pmatrix} \in \mathcal{M}_{\eta-j}(\mathbb{C}), \ \mathbf{B} = \begin{pmatrix} \frac{i^j}{\sqrt{2\pi}} \frac{\eta!}{j!} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{\eta-j},$$

and $a_{r\ell}$ given in (4.17). Again, following [AKBGBdT11a, Eq. (31),p. 570], it is possible to show

$$|P|_{\mathbb{C}^{\eta-j}} \leq C \left(\sum_{\substack{r\in[\![1,\eta-j-1]\!] \\ \ell\in[\![0,r]\!]}} \left| \Phi_k^{(\eta+r-\ell)}(i\Lambda_k) \right|^2 \right)^{\frac{\eta-j-1}{2}}. \tag{4.18}$$

Finally, let us estimate $|\Phi_k^{(\eta+r-\ell)}(i\Lambda_k)|$, for $r \in [\![1, \eta-j-1]\!]$ and $\ell \in [\![0, r]\!]$. Since Φ_k is an entire function, we can write

$$\Phi_k^{(m)}(i\Lambda_k) = \frac{m!}{2i\pi} \int_{|z-i\Lambda_k|=1} \frac{\Phi_k(z)}{(z-i\Lambda_k)^{m+1}}\, dz, \quad \forall m \geq 0,$$

so that

$$\left| \Phi_k^{(m)}(i\Lambda_k) \right| \leq C \sup_{z:|z-i\Lambda_k|=1} |\Phi_k(z)|.$$

Using inequality (4.12), the fact that $|z| \leq 1 + |\Lambda_k|$ for z such that $|z - i\Lambda_k| = 1$, inequalities (4.5), and the fact that $\tau_M < d^2/2$, we obtain

$$\left| \Phi_k^{(m)}(i\Lambda_k) \right| \leq C e^{C\sqrt{\Re(\Lambda_k)} + \frac{C}{\tau}}, \quad \forall k \geq 1, \ \forall m \geq 0.$$

Going back to (4.18), we get

$$|P|_{\mathbb{C}^{\eta-j}} \leq C e^{C\sqrt{\Re(\Lambda_k)} + \frac{C}{\tau}}.$$

Recall that the vector P contains the coefficients $p^{(r)}(i\Lambda_k)$ of the polynomial p. Thus, using that $|z|^r/r! \leq C e^{\frac{\eta}{4}\sqrt{|z|}}$ for any $r \in [\![0, \eta]\!]$, and using (4.5), we obtain

$$|p(z)| \leq C e^{\frac{\eta}{4}\sqrt{|z|} + C\sqrt{\Re(\Lambda_k)} + \frac{C}{\tau}}, \quad \forall z \in \mathbb{C}.$$

Combining the previous estimate, written for $x \in \mathbb{R}$, and (4.14) we deduce the expected bound (4.3) for $\Phi_{k,j} = p f_{k,j}$.

Appendix A. Proof of Proposition 4.3.

We start with another property satisfied by the sequence $\{\Lambda_k\}_{k\geq 1}$, namely that it behaves as k^2.

Lemma A.1. *Under assumptions $(\mathcal{H}_4)$, $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$, we have*

$$Ck \leq \sqrt{|\Lambda_k|} \leq C'k, \quad \forall k \geq 1. \tag{A.1}$$

The second lemma was often used.

Lemma A.2. *Let $\tau < d^2/2$. For the function M given by (4.8) we have $\tau_M < \tau$ (where τ_M is given in (4.9)).*

The next lemma are devoted to give bounds of every terms involved in the definitions (4.11) of W_k and $\widetilde{W}_k$.

Lemma A.3. *Under assumption* $(\mathcal{H}_6)$ *we have, for every* $z \in \mathbb{C}$ *and* $n \geq 1$,

$$\log|f(z)| \leq (d-1)\sqrt{|z|} + C, \quad \log|f_n(z)| \leq (d-1)\sqrt{|z|} + C,$$

where f *and* f_n *are defined in (4.6).*

Lemma A.4. *Under assumptions* $(\mathcal{H}_4)$, $(\mathcal{H}_5)$ *and* $(\mathcal{H}_6)$ *we have, for every* $n \geq 1$,

$$\log|f'(\Lambda_n)| \geq -C\sqrt{|\Lambda_n|},$$

where f *is defined in (4.6).*

Lemma A.5. *Let* $\tau < d^2/2$. *The function* M *given by* (4.8) *satisfies*

$$M(0) = 1, \quad \log|M(x)| \leq -d\sqrt{|x|} + \frac{C}{\tau}, \quad \forall x \in \mathbb{R}. \tag{A.2}$$

Lemma A.6. *Let* $\tau < d^2/2$. *The function* M *given by* (4.8) *satisfies*

$$\log|M(iy)| \geq 0, \quad \forall y \in \mathbb{R}, \tag{A.3}$$

and also

$$\log|M(iy)| \geq \tau_M|y| - C\sqrt{|y|} - \frac{C}{\tau}, \quad \forall y \in \mathbb{R}. \tag{A.4}$$

Proof of Proposition 4.3. Let us recall the definition of W_k:

$$W_k(z) = \frac{f(-iz)}{-if'(\Lambda_k)}\frac{M(z+\Im(\Lambda_k))}{M(i\Re(\Lambda_k))}.$$

From Lemma A.3 and A.4 and $|\Lambda_k| \leq C\Re(\Lambda_k)$ (see (4.5)) we deduce that

$$\left|\frac{f(-iz)}{-if'(\Lambda_k)}\right| \leq e^{(d-1)\sqrt{|z|}+C\sqrt{\Re(\Lambda_k)}}. \tag{A.5}$$

On the other hand, from inequality (A.4) of Lemma A.6 and using (4.10) we can also infer

$$\left|\frac{M(z+\Im(\Lambda_k))}{M(i\Re(\Lambda_k))}\right| \leq e^{\tau_M|z|+\tau_M(|\Im(\Lambda_k)|-\Re(\Lambda_k))+C\sqrt{\Re(\Lambda_k)}+\frac{C}{\tau}}.$$

Note that $\tau_M\,|\Im(\Lambda_k)| \leq C\sqrt{\Re(\Lambda_k)}$ thanks to $(\mathcal{H}_3)$ and $\tau_M < d^2/2$. Thus, putting both inequalities together we deduce estimate (4.12) for the function W_k.

Let us now take $x \in \mathbb{R}$. Applying inequality (A.2) of Lemma A.5 and, this time, inequality (A.3) of Lemma A.6, we arrive to

$$\left|\frac{M(x+\Im(\Lambda_k))}{M(i\Re(\Lambda_k))}\right| \leq e^{-d\sqrt{|x|}+d\sqrt{|\Im(\Lambda_k)|}+\frac{C}{\tau}}.$$

Note that $\sqrt{|\Im(\Lambda_k)|} \leq C\sqrt{\Re(\Lambda_k)}$ by (4.5). Thus, the previous inequality together with (A.5) (written for $x \in \mathbb{R}$) provide the estimate (4.13) for $W_k(x)$, with x real.

The same reasoning provide the estimate for $\widetilde{W}_k$.

Proof of Lemma A.1. The lower bound easily follows from $(\mathcal{H}_5)$ by taking $n = 1$.

To prove the upper bound, let us first observe that, for any k and n such that $|\Lambda_k| = |\Lambda_n|$, we have, using $(\mathcal{H}_4)$,

$$\left|\Re(\Lambda_k)^2 - \Re(\Lambda_n)^2\right| = \left|\Im(\Lambda_k)^2 - \Im(\Lambda_n)^2\right| \leq \beta^2 \left(\Re(\Lambda_k) + \Re(\Lambda_n)\right),$$

so that

$$|\Re(\Lambda_k) - \Re(\Lambda_n)| \leq \beta^2.$$

It follows that (using $(\mathcal{H}_4)$ again)

$$|\Lambda_k - \Lambda_n| \leq |\Re(\Lambda_k) - \Re(\Lambda_n)| + |\Im(\Lambda_k) - \Im(\Lambda_n)| \leq \beta^2 + 2\beta\sqrt{|\Lambda_k|}.$$

By using $(\mathcal{H}_5)$, and the fact that $k + n \geq k$, we obtain

$$|k - n| \leq \max\left\{q, \frac{\beta^2 + 2\beta\sqrt{|\Lambda_k|}}{\rho k}\right\}.$$

Note that if k is such that $\frac{\beta^2+2\beta\sqrt{|\Lambda_k|}}{\rho k} \leq q$ then $\sqrt{|\Lambda_k|} \leq \left(\frac{q\rho}{2\beta}\right) k$ and we are done. Let us then deal with the k such that $\frac{\beta^2+2\beta\sqrt{|\Lambda_k|}}{\rho k} > q$.

Applying the previous estimate with $n = \mathcal{N}(|\Lambda_k|)$ (which indeed satisfies $|\Lambda_n| = |\Lambda_k|$ by (3.5) and $(\mathcal{H}_4)$), we deduce that

$$\mathcal{N}(|\Lambda_k|) \leq k + |\mathcal{N}(|\Lambda_k|) - k| \leq k + \frac{\beta^2 + 2\beta\sqrt{|\Lambda_k|}}{\rho k},$$

and by $(\mathcal{H}_6)$ we finally obtain

$$p\sqrt{|\Lambda_k|} \leq \alpha + \mathcal{N}(|\Lambda_k|) \leq k + \frac{\beta^2 + 2\beta\sqrt{|\Lambda_k|}}{\rho k}.$$

For k large enough, we obtain

$$\frac{p}{2}\sqrt{|\Lambda_k|} \leq k + \frac{\beta^2}{\rho k} \leq \left(1 + \frac{\beta^2}{\rho}\right) k,$$

and the lemma is proved.

Proof of Lemma A.2. For the proof we will follow some ideas from [FR71] and [Mil04] (see also [Red77]). Let us consider the counting function N associated with the sequence $\{a_n\}_{n\geq 1}$ given by (4.7):

$$N(r) = \#\{n \geq 1 : a_n \leq r\}.$$

Observe that the sequence $\{a_n\}_{n\geq 0}$ can be written as

$$a_n = a_0 + \frac{n^2}{A^2}, \quad \forall n \geq 1, \quad \text{with} \quad A = \frac{d}{2} \quad \text{and} \quad a_0 = \frac{d^2}{\tau^2} - \frac{4}{d^2},$$

and that $a_0 > 0$ since we assumed that $\tau < d^2/2$. Thus, $N(r) = 0$ for $r < a_1$, and

$$N(r) = \lfloor A\sqrt{r - a_0} \rfloor, \quad \forall r \geq a_1,$$

where we recall that $\lfloor\cdot\rfloor$ is the floor function. Note that

$$A\sqrt{r} - A\sqrt{a_0} \leq N(r) \leq A\sqrt{r}, \quad \forall r \geq 0.$$

These remarks in mind, we have

$$\begin{aligned}\tau_M = \sum_{n\geq 1} \frac{1}{a_n} &= \int_{a_1^-}^{+\infty} \frac{1}{r}\, dN(r) = \int_{a_1^-}^{+\infty} \frac{N(r)}{r^2}\, dr \leq \int_{a_1}^{+\infty} \frac{A\sqrt{r - a_0}}{r^2}\, dr \\ &< A\int_{a_1}^{+\infty} \frac{\sqrt{r}}{r^2}\, dr = \frac{2A}{\sqrt{a_1}} = \tau,\end{aligned}$$

where the last inequality is strict since $a_0 \neq 0$.

Proof of Lemma A.3. Given $z \in \mathbb{C}$, one has

$$\log|f(z)| \leq \sum_{k\geq 1} \log\left(1+\frac{|z|}{|\Lambda_k|}\right) = \int_{|\Lambda_1|^-}^{+\infty} \log\left(1+\frac{|z|}{t}\right) d\mathcal{N}(t).$$

Taking into account $\lim_{t\to+\infty} \mathcal{N}(t)/t = 0$ (consequence of $(\mathcal{H}_6)$) an integration by parts gives

$$\int_{|\Lambda_1|^-}^{+\infty} \log\left(1+\frac{|z|}{t}\right) d\mathcal{N}(t) = \int_{|\Lambda_1|^-}^{+\infty} \frac{|z|}{t(|z|+t)}\mathcal{N}(t)\, dt.$$

After the change of variable $t = |z|s$, we obtain

$$\int_{|\Lambda_1|^-}^{+\infty} \frac{|z|}{t(|z|+t)}\mathcal{N}(t)\, dt = \int_{|\Lambda_1|^-/|z|}^{+\infty} \frac{\mathcal{N}(|z|s)}{s(s+1)}\, ds.$$

From $(\mathcal{H}_6)$, we conclude that

$$\begin{aligned}\int_{|\Lambda_1|^-/|z|}^{+\infty} \frac{\mathcal{N}(|z|s)}{s(s+1)}\, ds &\leq p\sqrt{|z|}\int_{|\Lambda_1|^-/|z|}^{+\infty} \frac{1}{\sqrt{s}(s+1)}\, ds + \alpha \int_{|\Lambda_1|^-/|z|}^{+\infty} \frac{1}{s(s+1)}\, ds \\ &\leq p\pi\sqrt{|z|} + \alpha \log\left(1+\frac{|z|}{|\Lambda_1|}\right).\end{aligned}$$

Since the function $z \in \mathbb{C} \longmapsto \alpha\log(1+|z|/\,|\Lambda_1|) - \sqrt{|z|}$ is bounded on $\mathbb{C}$, the lemma is proved. Repeating the arguments, we obtain the same estimate for f_n.

Proof of Lemma A.4. For proving the result we are going to follow some ideas from [LK71] and [FR75] (see also [FCGBdT10]).

Firstly, note that

$$f'(\Lambda_n) = -\frac{1}{\Lambda_n}\prod_{k\neq n}\left(1-\frac{\Lambda_n}{\Lambda_k}\right), \quad \forall n \geq 1. \tag{A.6}$$

Given $n \geq 1$, let us introduce the sets

$$S_1(n) = \{k \neq n : |\Lambda_k| \leq 2|\Lambda_n|\} \quad \text{and} \quad S_2(n) = \{k : |\Lambda_k| > 2|\Lambda_n|\}.$$

and the infinite product

$$\mathcal{P}_n = \prod_{k\neq n}\left|1-\frac{\Lambda_n}{\Lambda_k}\right|. \tag{A.7}$$

Let us give a lower bound for the product $\mathcal{P}_n$. To this end, we split this product into two parts using the sets $S_1(n)$ and $S_2(n)$:

1. From the definition of $S_1(n)$ and using $(\mathcal{H}_5)$, we can write

$$\prod_{k\in S_1(n)}\left|1-\frac{\Lambda_n}{\Lambda_k}\right| = \prod_{\substack{k\in S_1(n)\\ |k-n|\geq q}}\left|\frac{\Lambda_k-\Lambda_n}{\Lambda_k}\right| \prod_{\substack{k\in S_1(n)\\ |k-n|< q}}\left|\frac{\Lambda_k-\Lambda_n}{\Lambda_k}\right| \geq \prod_{\substack{k\in S_1(n)\\ |k-n|\geq q}}\frac{\rho}{2}\frac{|k-n|(k+n)}{|\Lambda_n|} \prod_{\substack{k\in S_1(n)\\ |k-n|< q}}\frac{1}{2}\frac{A}{|\Lambda_n|},$$

where

$$A = \inf_{k\neq n: |k-n|<q} |\Lambda_k - \Lambda_n| > 0.$$

It follows that

$$\prod_{k\in S_1(n)}\left|1-\frac{\Lambda_n}{\Lambda_k}\right| \geq \prod_{k\in S_1(n)}\frac{\rho}{2}\frac{|k-n|(k+n)}{|\Lambda_n|} \prod_{\substack{k\in S_1(n)\\ |k-n|< q}}\frac{A}{\rho|k-n|(k+n)}.$$

Since

$$\prod_{\substack{k\in S_1(n)\\|k-n|<q}} \frac{A}{\rho|k-n|} \geq \left(\frac{A}{\rho q}\right)^{2q-1}, \quad \prod_{\substack{k\in S_1(n)\\|k-n|<q}} \frac{1}{k+n} \geq \frac{1}{(2n+q-1)^{2q-1}}, \quad \forall n\geq 1,$$

we deduce that

$$\prod_{\substack{k\in S_1(n)\\|k-n|<q}} \frac{A}{|k-n|(k+n)} \geq \frac{C}{(2n+q-1)^{2q-1}}.$$

As $|\Lambda_n| \geq Cn^2$ for every $n\geq 1$ (see (A.1)), we obtain

$$\prod_{\substack{k\in S_1(n)\\|k-n|<q}} \frac{A}{|k-n|(k+n)} \geq \frac{C}{|\Lambda_n|^{\frac{2q-1}{2}}}.$$

Let us define $r_n = \#\{k\in S_1(n) : k<n\}$ and $s_n = \#\{k\in S_1(n) : k>n\}$. From (A.1), we deduce that $k+n\geq C\sqrt{|\Lambda_n|}$ for any $n,k\geq 1$. Thus,

$$\prod_{k\in S_1(n)} \left|1-\frac{\Lambda_n}{\Lambda_k}\right| \geq C\,|\Lambda_n|^{-q-\frac{1}{2}}\, r_n! \left(\frac{\rho\gamma_2}{2|\Lambda_n|^{1/2}}\right)^{r_n} s_n! \left(\frac{\rho\gamma_2}{2|\Lambda_n|^{1/2}}\right)^{s_n} = C\,|\Lambda_n|^{-q-\frac{1}{2}}\,\mathcal{P}_n^{(1)}\mathcal{P}_n^{(2)}, \quad \forall n\geq 1. \tag{A.8}$$

Let us argue with $\mathcal{P}_n^{(1)}$. A similar reasoning will provide a lower bound for $\mathcal{P}_n^{(2)}$.

Observe that there exists two constants $c_0, c_1>0$ such that

$$r! \geq c_0\left(\frac{r}{e}\right)^r, \quad \forall r\geq 1,$$

and

$$-c_1 = \inf_{s>0} s(\log s).$$

We can then write

$$\begin{aligned}\mathcal{P}_n^{(1)} = r_n!\left(\frac{\rho\gamma_2}{2|\Lambda_n|^{1/2}}\right)^{r_n} &\geq c_0\left(\frac{\rho\gamma_2 r_n}{2e|\Lambda_n|^{1/2}}\right)^{r_n}\\ &= c_0\exp\left[\frac{2e|\Lambda_n|^{1/2}}{\rho\gamma_2}\left(\frac{\rho\gamma_2 r_n}{2e|\Lambda_n|^{1/2}}\right)\log\left(\frac{\rho\gamma_2 r_n}{2e|\Lambda_n|^{1/2}}\right)\right] \geq c_0\exp\left(-\frac{2ec_1}{\rho\gamma_2}|\Lambda_n|^{1/2}\right).\end{aligned}$$

Putting this inequality (and the similar one for the product $\mathcal{P}_n^{(2)}$) in (A.8) we obtain

$$\prod_{k\in S_1(n)} \left|1-\frac{\Lambda_n}{\Lambda_k}\right| \geq e^{-C\sqrt{|\Lambda_n|}-C}, \quad \forall n\geq 1. \tag{A.9}$$

2. Let us now estimate the product (A.7) for $k\in S_2(n)$ that we denote by $\mathcal{P}_n^{(3)}$. Let $c_2>0$ be such that

$$\log(1-s) \geq -c_2 s, \quad \forall s\in[0,1/2]. \tag{A.10}$$

Observe that, for $k \in S_2(n)$ one has $|\Lambda_n|/|\Lambda_k| \leq 1/2$, so that we can use (A.10) to obtain

$$\begin{aligned}
\log \mathcal{P}_n^{(3)} &\geq \sum_{k \in S_2(n)} \log\left(1 - \frac{|\Lambda_n|}{|\Lambda_k|}\right) \geq -c_2|\Lambda_n| \sum_{k\in S_2(n)} \frac{1}{|\Lambda_k|} = -c_2|\Lambda_n| \int_{2|\Lambda_n|} \frac{1}{r} d\mathcal{N}(r) \\
&= -c_2|\Lambda_n| \left(-\frac{\mathcal{N}(2|\Lambda_n|)}{2|\Lambda_n|} + \int_{2|\Lambda_n|^-} \frac{\mathcal{N}(r)}{r^2} dr\right) \geq -c_2|\Lambda_n| \int_{2|\Lambda_n|^-} \frac{\mathcal{N}(r)}{r^2} dr \\
&\geq -c_2|\Lambda_n| \int_{2|\Lambda_n|^-} \frac{\alpha + p\sqrt{r}}{r^2} dr = -c_2|\Lambda_n| \left(\frac{\alpha}{2|\Lambda_n|} + \frac{2p}{\sqrt{2|\Lambda_n|}}\right) \\
&= -\frac{\alpha c_2}{2} - \sqrt{2} p c_2 |\Lambda_n|^{1/2}.
\end{aligned}$$

Putting (A.9) and this last inequality in (A.7), we deduce

$$\mathcal{P}_n = \prod_{k \neq n} \left|1 - \frac{\Lambda_n}{\Lambda_k}\right| \geq e^{-C\sqrt{|\Lambda_n|} - C}, \quad \forall n \geq 1,$$

Since $|\Lambda_n| \geq |\Lambda_1|$ for every $n \geq 1$ (see $(\mathcal{H}_4)$) we finally have

$$\mathcal{P}_n \geq e^{-C\sqrt{|\Lambda_n|}}, \quad \forall n \geq 1.$$

This inequality and formula (A.6) provide the desired estimate. This ends the proof.

Proof of Lemma A.5. For the proof we will follow some ideas from [FR71] and [Mil04] (see also [Red77]). Let us first consider again the counting function N associated with the sequence $\{a_n\}_{n\geq 1}$ given by (4.7):

$$N(r) = \#\{n \geq 1 : a_n \leq r\}.$$

Observe again that the sequence $\{a_n\}_{n\geq 0}$ can be written as

$$a_n = a_0 + \frac{n^2}{A^2}, \quad \forall n \geq 1, \quad \text{with} \quad A = \frac{d}{2} \quad \text{and} \quad a_0 = \frac{d^2}{\tau^2} - \frac{4}{d^2}, \tag{A.11}$$

and that $a_0 > 0$ since we assumed that $\tau < d^2/2$. Thus, $N(r) = 0$ for $r < a_1$, and

$$N(r) = \lfloor A\sqrt{r - a_0} \rfloor, \quad \forall r \geq a_1, \tag{A.12}$$

We will often use that

$$A\sqrt{r} - A\sqrt{a_0} \leq N(r) \leq A\sqrt{r}, \quad \forall r \geq 0.$$

Let us prove the inequality (A.2). Observe that M is an even function. So, we will show (A.2) for $x \in (0, +\infty)$. From the definition (4.8) of M, one has

$$\log|M(x)| = \sum_{n\geq 1} \log\left|\frac{\sin(x/a_n)}{x/a_n}\right| = \int_{a_1^-}^{+\infty} g\left(\frac{x}{r}\right) dN(r),$$

here

$$g(s) = \log\left|\frac{\sin s}{s}\right|, \quad s \in \mathbb{R}.$$

- Since, g is non increasing on $[0, 1)$, for any $x \in [0, a_1]$, we have

$$\log|M(x)| \leq \log|M(0)| = 0 \leq -d\sqrt{x} + d\sqrt{a_1} \leq -d\sqrt{x} + \frac{d^2}{\tau},$$

which gives the claim in that case.

- Assume now that $x > a_1$. We write

$$\log|M(x)| = \sum_{a_n \le x} g(x/a_n) + \sum_{a_n > x} g(x/a_n) \equiv I + J.$$

Since g is negative and non increasing on $[0,1]$, the second sum J can be bounded as follows

$$\begin{aligned} J &\le \sum_{2x \ge a_n > x} g(x/a_n) \le -|g(1/2)|(N(2x) - N(x)) \\ &\le -|g(1/2)|(A\sqrt{2x-a_0} - 1 - A\sqrt{x-a_0}) = |g(1/2)| - A|g(1/2)|\frac{x}{\sqrt{2x-a_0}+\sqrt{x-a_0}} \\ &\le |g(1/2)| - A\frac{|g(1/2)|}{\sqrt{2}+1}\sqrt{x}. \end{aligned}$$

In the first sum I, we use the inequality $g(s) \le -\log s$ for any $s \ge 0$, to get

$$\begin{aligned} I &\le -\sum_{a_n \le x} \log(x/a_n) = \int_{a_1^-}^{x} \log\left(\frac{r}{x}\right) dN(r) = -\int_{a_1}^{x} \frac{N(r)}{r}\, dr \\ &\le \int_{a_1}^{x} \frac{1 - A\sqrt{r-a_0}}{r}\, dr = \log(x/a_1) - A\left(\int_{a_1}^{x} \frac{1}{\sqrt{r-a_0}}\, dr - a_0\int_{a_1}^{x} \frac{1}{r\sqrt{r-a_0}}\, dr\right) \\ &\le \log(x/a_1) - 2A\sqrt{x-a_0} + 2A\sqrt{a_1-a_0} + A\sqrt{a_0}\int_1^{+\infty} \frac{1}{r\sqrt{r-1}}\, dr \\ &\le -2A\sqrt{x} + c_1A\sqrt{a_0} + \log(x) + 2, \end{aligned}$$

with $c_1 = 2 + \int_1^{+\infty} \frac{1}{r\sqrt{r-1}}\, dr$.
Combining the two estimates gives

$$\log|M(x)| \le -A\left(2 + \frac{|g(1/2)|}{1+\sqrt{2}}\right)\sqrt{x} + \log x + c_1A\sqrt{a_0} + 2 + |g(1/2)|.$$

Observe now that $a_0 \le d^2/\tau^2$, that $2A = d$ and that the function

$$x \in [0, +\infty[\mapsto -A\frac{|g(1/2)|}{1+\sqrt{2}}\sqrt{x} + \log(x) + 2 + |g(1/2)|,$$

is bounded by some number $c_2 > 0$ depending only on $A = d/2$. We finally get the inequality

$$\log|M(x)| \le -d\sqrt{x} + \frac{c_1d^2}{2\tau} + c_2,$$

which gives the claim by using that $1 \le \frac{d^2}{2\tau}$.

Proof of Lemma A.6. We start by observing that

$$\frac{\sin iy}{iy} = \frac{\sinh y}{y} \ge 1, \quad \forall y \in \mathbb{R}.$$

As a consequence, we obtain $M(iy) \ge 1$, for any $y \in \mathbb{R}$. Thus, we immediately get (A.3).

We will now obtain the proof of (A.4) by adapting the proof of Lemma 6.3 of [FR71] to the sequence $\{a_n\}_{n\ge1}$ given by (4.7). We set $c_0 = \log\sqrt{3} > 0$.

- Assume first that $|y|/c_0 \le a_1$. Then, by using (A.3), we get

$$\log|M(iy)| \ge 0 \ge \tau_M|y| - \tau_Mc_0a_1 = \tau_M|y| - \tau_Mc_0\frac{d^2}{\tau^2} \ge \tau_M|y| - c_0\frac{d^2}{\tau}, \quad \forall \frac{|y|}{c_0} \le a_1 = \frac{d^2}{\tau^2},$$

and the claim is proved in that case.

- Assume now that $|y|/c_0 \geq a_1$. Observe that

$$\frac{\sin(iy)}{iy} = \frac{1}{2}\left(\frac{e^y - e^{-y}}{y}\right) = \frac{e^{|y|} - e^{-|y|}}{2|y|} = e^{|y|}\frac{1 - e^{-2|y|}}{2|y|}, \quad \forall y \neq 0.$$

Thus, using the definitions (4.8) and (4.9) of M and τ_M, we have

$$\log|M(iy)| = \sum_{n\geq 1}\frac{|y|}{a_n} + \sum_{n\geq 1}\log\left(\frac{1 - e^{-2|y|/a_n}}{2|y|/a_n}\right) = \tau_M|y| + \mathcal{I}, \tag{A.13}$$

where the sequence $\{a_n\}_{n\geq 1}$ is given by (4.7).
In order to bound the series $\mathcal{I}$, we will use the inequalities

$$\frac{1 - e^{-2y}}{2y} \geq e^{-2y}, \quad \forall y > 0, \quad \text{and} \quad \frac{1 - e^{-2y}}{2y} \geq \frac{1}{3y}, \quad \forall y \geq \log\sqrt{3} = c_0.$$

So, for $y \in \mathbb{R}$ with $|y|/c_0 \geq a_1$, one has,

$$\mathcal{I} = \sum_{n\geq 1}\log\left(\frac{1 - e^{-2|y|/a_n}}{2|y|/a_n}\right) \geq -\sum_{\substack{n\geq 1\\ a_n > |y|/c_0}}\frac{2|y|}{a_n} + \sum_{\substack{n\geq 1\\ a_n \leq |y|/c_0}}\log\left(\frac{a_n}{3|y|}\right) \equiv \mathcal{I}_1 + \mathcal{I}_2\,. \tag{A.14}$$

  - Let us first bound from below $\mathcal{I}_1$ in the expression (A.14). One has

$$\mathcal{I}_1 = -\sum_{\substack{n\geq 1\\ a_n > |y|/c_0}}\frac{2|y|}{a_n} = -2|y|\int_{a_{n_0}^-}^{+\infty}\frac{dN(r)}{r} \geq -2|y|\int_{|y|/c_0}^{+\infty}\frac{dN(r)}{r},$$

  where $n_0 \geq 1$ is the smallest integer such that $a_{n_0} > |y|/c_0$ and $N(\cdot)$ is the counting function associated to the sequence $\{a_n\}_{n\geq 1}$ (see (A.11) and (A.12)). Integrating by parts, we obtain:

$$\begin{aligned}\mathcal{I}_1 &\geq -2|y|\left[\frac{1}{r}N(r)\Big|_{|y|/c_0}^{+\infty} + \int_{|y|/c_0}^{+\infty}\frac{N(r)}{r^2}\,dr\right] \geq -2|y|A\int_{|y|/c_0}^{+\infty}\frac{\sqrt{r - a_0}}{r^2}\,dr\\ &\geq -2A|y|\int_{|y|/c_0}^{+\infty} r^{-3/2}\,dr,\end{aligned}$$

  that is to say,

$$\mathcal{I}_1 \geq -4c_0^{1/2}A\sqrt{|y|}, \quad \forall\frac{|y|}{c_0} > a_1\,. \tag{A.15}$$

  - Let us deal with the second term $\mathcal{I}_2$ in (A.14) for $|y|$ satisfying $a_1 < |y|/c_0$. Using that for any $r \in [a_1, |y|/c_0]$ one has $r < 3|y|$ ($c_0 = \log\sqrt{3}$), we can write

$$\mathcal{I}_2 = \sum_{\substack{n\geq 1\\ a_n \leq |y|/c_0}}\log\left(\frac{a_n}{3|y|}\right) = \int_{a_1^-}^{a_{n_1}}\log\left(\frac{r}{3|y|}\right)dN(r) \geq \int_{a_1^-}^{|y|/c_0}\log\left(\frac{r}{3|y|}\right)dN(r),$$

  where $n_1 \geq 1$ is the largest integer such that $a_{n_1} \leq |y|/c_0$.
  Again, integrating by parts, we deduce

$$\begin{aligned}\mathcal{I}_2 &\geq N(r)\log\left(\frac{r}{3|y|}\right)\Big|_{a_1^-}^{|y|/c_0} - \int_{a_1^-}^{|y|/c_0}\frac{N(r)}{r}\,dr = -\log(3c_0)N(|y|/c_0) - A\int_{a_1^-}^{|y|/c_0}\frac{\sqrt{r - a_0}}{r}\,dr\\ &\geq -\log(3c_0)N(|y|/c_0) - A\int_0^{|y|/c_0}\frac{1}{\sqrt{r}}\,dr \geq -A\left(2 + \log(3c_0)\right)c_0^{-1/2}\sqrt{|y|}.\end{aligned}$$

In view of (A.13) and (A.14), this last inequality together with (A.15) provide ($c_0 = \log\sqrt{3}$)

$$\log|M(iy)| \geq \tau_M|y| - c_1 d\sqrt{|y|},$$

with $c_1 = (1 + 2c_0 + \log(3c_0)/2)c_0^{-1/2}$.

Owing to the previous calculations, we finally obtain the inequality (A.4). This ends the proof.

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